

Riemann Sums key (Practice Problems) Exact.

1. $\int_0^2 2x-5 dx$
a.

① $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ ② $\Delta x_i = 0 + \frac{2i}{n} = \frac{2i}{n}$

③ $f(x_i) = 2\left(\frac{2i}{n}\right) - 5 = \frac{4i}{n} - 5$

④ $\sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(\frac{4i}{n} - 5\right) \frac{2}{n} = \sum_{i=1}^n \left(\frac{8i}{n^2} - \frac{10}{n}\right)$

⑤ $\lim_{n \rightarrow \infty} \sum \left(\frac{8i}{n^2} - \frac{10}{n}\right) = \sum \frac{8i}{n^2} - \sum \frac{10}{n} =$

$\frac{8}{n^2} \sum i - \sum \frac{10}{n} = \frac{8}{n^2} \left[\frac{n(n+1)}{2}\right] - \frac{10}{n} \cdot n =$

$\frac{8n}{2n} + \frac{4}{2n} - 10 = 4 + \frac{4}{n} - 10 = -6 + \frac{4}{n}$

⑥ $\lim_{n \rightarrow \infty} -6 + \frac{4}{n} = -6$

by Fundamental Theorem: $\int_0^2 2x-5 dx = x^2 - 5x \Big|_0^2 = 4 - 10 = -6 \checkmark$

b. $\int_1^4 2x-5 dx$

① $\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$ ② $\Delta x_i = 1 + \frac{3i}{n}$

③ $f(x_i) = 2\left(1 + \frac{3i}{n}\right) - 5 = 2 + \frac{6i}{n} - 5 = \frac{6i}{n} - 3$

④ $\sum_{i=1}^n f(x_i) = \sum_{i=1}^n \left(\frac{6i}{n} - 3\right) \frac{3}{n} = \sum_{i=1}^n \left(\frac{12i}{n^2} - \frac{9}{n}\right)$

⑤ $\sum_{i=1}^n \left(\frac{12i}{n^2} - \frac{9}{n}\right) = \sum_{i=1}^n \frac{12i}{n^2} - \sum_{i=1}^n \frac{9}{n} = \frac{12}{n^2} \sum_{i=1}^n i - \sum_{i=1}^n \frac{9}{n}$

$= \frac{12}{n^2} \left(\frac{n(n+1)}{2}\right) - \frac{9}{n} \cdot n = \frac{6n}{n} + \frac{6}{n} - 9 =$

$6 + \frac{6}{n} - 9 = -3 + \frac{6}{n}$

⑥ $\lim_{n \rightarrow \infty} -3 + \frac{6}{n} = -3$

by FTC: $\int_1^4 2x-5 dx = x^2 - 5x \Big|_1^4 = (16 - 20) - (1 - 5) = -4 - (-4) = 0$

$$2. a. \int_0^2 x^2 + 1 dx$$

$$\textcircled{1} \Delta x = \frac{2}{n} \quad \textcircled{2} x_i = a + \Delta x i = 0 + \frac{2i}{n} = \frac{2i}{n}$$

$$\textcircled{3} f(x_i) = \left(\frac{2i}{n}\right)^2 + 1 = \frac{4i^2}{n^2} + 1$$

$$\textcircled{4} \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(\frac{4i^2}{n^2} + 1\right) \left(\frac{2}{n}\right) = \sum_{i=1}^n \left(\frac{8i^2}{n^3} + \frac{2}{n}\right)$$

$$\textcircled{5} \frac{8i}{n^3} \sum_{i=1}^n i^2 + \sum_{i=1}^n \frac{2}{n} = \frac{8i}{n^3} \left(\frac{n(2n+1)(n+1)}{6}\right) + \frac{2}{n} \cdot n$$

$$= \frac{4(2n^2 + 3n + 1)}{3n^2} + 2 = \frac{8n^2}{3n^2} + \frac{4 \cdot 2n}{3n^2} + \frac{4}{3n^2} + 2$$

$$\textcircled{6} \lim_{n \rightarrow \infty} \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} + 2 = \frac{8}{3} + 2 = \frac{14}{3}$$

$$\text{by FTC: } \int_0^2 x^2 + 1 dx = \left. \frac{1}{3}x^3 + x \right|_0^2 = \frac{1}{3}(8) + 2 = \frac{14}{3} \checkmark$$

$$b. \int_1^4 x^2 + 1 dx$$

$$\textcircled{1} \Delta x = \frac{4-1}{n} = \frac{3}{n} \quad \textcircled{2} x_i = 1 + \frac{3i}{n}$$

$$\textcircled{3} f(x_i) = \left(1 + \frac{3i}{n}\right)^2 + 1 = 1 + \frac{6i}{n} + \frac{9i^2}{n^2} + 1 = 2 + \frac{6i}{n} + \frac{9i^2}{n^2}$$

$$\textcircled{4} \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(2 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) \frac{3}{n} =$$

$$\sum_{i=1}^n \left(\frac{6}{n} + \frac{18i}{n^2} + \frac{27i^2}{n^3}\right)$$

$$\textcircled{5} = \sum_{i=1}^n \frac{6}{n} + \frac{18}{n^2} \sum_{i=1}^n i + \frac{27}{n^3} \sum_{i=1}^n i^2 =$$

$$\frac{6}{n} \cdot n + \frac{18}{n^2} \left[\frac{n(n+1)}{2}\right] + \frac{27}{n^3} \left[\frac{(2n+1)(n+1)(n)}{6}\right]$$

$$= 6 + \frac{9n}{n} + \frac{9}{n} + \frac{9}{2n^2}(2n^2 + 3n + 1) =$$

$$6 + 9 + \frac{9}{n} + \frac{18n^2}{2n^2} + \frac{27n}{2n^2} + \frac{9}{2n^2} =$$

$$6 + 9 + \frac{9}{n} + 9 + \frac{27}{2n} + \frac{9}{2n^2} = 24 + \frac{36}{2n} + \frac{9}{2n^2}$$

$$\textcircled{6} \lim_{n \rightarrow \infty} 24 + \frac{18}{n} + \frac{9}{2n^2} = 24$$

$$\text{by FTC: } \int_1^4 x^2 + 1 dx = \left. \frac{1}{3}x^3 + x \right|_1^4 = \frac{1}{3}(64) + 4 - \left(\frac{1}{3}(1) + 1\right) =$$

$$\frac{64-1}{3} + 3 = \frac{63}{3} + 3 = 21 + 3 = 24 \checkmark$$

③ a. $\int_0^2 x^2 - x + 6 dx$

① $\Delta x = \frac{2}{n}$ $\Delta x_i = \frac{2i}{n}$

② $f(x_i) = \left(\frac{2i}{n}\right)^2 - \frac{2i}{n} + 6 = \frac{4i^2}{n^2} - \frac{2i}{n} + 6$

④ $\sum_{i=1}^n \left(\frac{4i^2}{n^2} - \frac{2i}{n} + 6\right) \left(\frac{2}{n}\right) =$
 $\sum_{i=1}^n \left(\frac{8i^2}{n^2} - \frac{4i}{n^2} + \frac{12}{n}\right)$

⑤ $\frac{8}{n^3} \sum i^2 - \frac{4}{n^2} \sum i + \frac{12}{n} \sum 1 =$
 $\frac{8}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] - \frac{4}{n^2} \left[\frac{n(n+1)}{2} \right] + \frac{12}{n} \cdot n$
 $= \frac{4(2n^2 + 3n + 1)}{3n^2} - \frac{2n+1}{n} + 12 =$
 $\frac{8n^2}{3n^2} + \frac{4n}{3n^2} + \frac{4}{3n^2} - \frac{2n}{n} - \frac{2}{n} + 12 =$
 $\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2} - 2 - \frac{2}{n} + 12 =$
 $\frac{38}{3} + \frac{2}{n} + \frac{4}{3n^2}$

⑥ $\lim_{n \rightarrow \infty} \left(\frac{38}{3} + \frac{2}{n} + \frac{4}{3n^2} \right) = \frac{38}{3}$

by FTC $\int_0^2 x^2 - x + 6 dx = \left. \frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x \right|_0^2 = \frac{8}{3} - 2 + 12 = \frac{38}{3} \checkmark$

b) $\int_1^4 x^2 - x + 6 dx$

① $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ② $\Delta x_i = a + \Delta x_i = 1 + \frac{3i}{n}$

③ $f(x_i) = \left(1 + \frac{3i}{n}\right)^2 - \left(1 + \frac{3i}{n}\right) + 6 =$
 $1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 1 - \frac{3i}{n} + 6 = 6 + \frac{3i}{n} + \frac{9i^2}{n^2}$

④ $\sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(6 + \frac{3i}{n} + \frac{9i^2}{n^2}\right) \left(\frac{3}{n}\right) =$
 $\sum_{i=1}^n \left(\frac{18}{n} + \frac{9i}{n^2} + \frac{27i^2}{n^3}\right)$

⑤ $\frac{18}{n} \sum 1 + \frac{9}{n^2} \sum i + \frac{27}{n^3} \sum i^2 =$
 $\frac{18}{n} \cdot n + \frac{9}{n^2} \left[\frac{n(n+1)}{2} \right] + \frac{27}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right]$
 $18 + \frac{9n}{2n} + \frac{9}{2n} + \frac{9(2n^2 + 3n + 1)}{2n^2} =$
 $18 + \frac{9}{2} + \frac{9}{2n} + \frac{18n^2}{2n^2} + \frac{27n}{2n^2} + \frac{9}{2n^2} =$
 $\frac{63}{2} + \frac{36}{2n} + \frac{9}{2n^2}$

⑥ $\lim_{n \rightarrow \infty} \left(\frac{63}{2} + \frac{18}{n} + \frac{9}{2n^2} \right) = \frac{63}{2}$

by FTC $\int_1^4 x^2 - x + 6 dx = \left. \frac{1}{3}x^3 - \frac{1}{2}x^2 + 6x \right|_1^4 =$
 $\frac{1}{3}(64) - \frac{1}{2}(16) + 24 - \left(\frac{1}{3}(1) - \frac{1}{2}(1) + 6 \right) = \frac{64}{3} + 16 - \frac{1}{3} + \frac{1}{2} - 6 = \frac{63}{2} \checkmark$