SOLVING QUADRATIC EQUATIONS

One of the first applications of the factoring techniques we've been learning is to use it to solve quadratic equations with it. Everything we do here is based on a property we learned way back in the beginning of algebra when we were learning properties of real numbers.

Zero Product Property
I f $a \cdot b = 0$, then either a=0 or b=0.

This property says, essentially, that 0 has a special property that belongs to no other number. It's that if two numbers are multiplied together to get zero, then one of those two numbers must be zero itself. The only way to get a zero, is to have a zero.

This is not true of any other number. Two numbers multiplied together to get 1 doesn't not tell you anything about the numbers except that they are reciprocals of each other. They might both be one, or one could be 2 and the other ½. We don't know. Only with zero can we conclude anything about the two numbers, and that's why solving for zero in every single problem will be an essential component of our process. None of our factoring will mean anything unless one side of the equation is zero.

Let's look at the steps for our process.

Step 1. If you have terms on both sides of the equation, you must simplify and combine like terms. If not, skip to step 3.

Step 2. Move everything to one side of the equation so that the other side is zero.

Step 3. Factor. If everything was already factored in step 1, you don't need to FOIL everything out unless there are no common factors. If you can't factor at this step, return to step 1 and simplify. [Hint: problems with more than two factors are usually not meant to be FOILed but worked with as is.]

Step 4. Using the Zero Product Property, set each factor equal to zero.

Step 5. Solve each (usually linear) equation produced in Step 4.

Step 6. List the set of solutions you found from each equation in Step 5.

We'll start by doing some simpler examples that start at Step 4, and then we'll do some longer examples that require all 6 steps.

Example 1. $(x+3)(2x-1)=0$
This equation is already factored, and the right side is equal to zero, so we've met all the conditions in steps 2 and 3 already. The next thing is to set each factor equal to zero.

$$x+3=0 \rightarrow x=-3$$
$$2x-1=0 \rightarrow 2x=1 \rightarrow x=\frac{1}{2}$$

Here, I've set each factor equal to zero. Each one is linear, so we can easily solve for x. In the first equation, I subtracted 3 from both sides. In the second, I needed two steps. First, I added 1 to both sides, and then I divided by 2. I'm using the arrows here to indicate the next step. I can't use equal signs as I do for expressions because there is already an equal sign in the problem. My solution then, for Step 6 is $x=\left\{-3,\frac{1}{2}\right\}$. You'll often see the set of solutions listed like this in set notation brackets.

Example 2. $(b-1)(b+2)(b-3)=0$
It's common for our problems to have two solutions, a little less common to have three, though it is possible. The trick with higher-order polynomials is in the factoring. But when we are given the factored form, it doesn't matter how many of them there are. Here, we have one side equal zero and the left side factored, so we can move on to Step 4 and solve.

$$b-1=0 \rightarrow b=1$$
$$b+2=0 \rightarrow b=-2$$
$$b-3=0 \rightarrow b=3$$

So, $b=\{1,-2,3\}$.

Example 3. $10a^2-5a-15=0$
Now, we still have the conditions for Step 2 satisfied, but the equation isn't factored. Let's do that. Remember, start with the GCF (which is 5 here) and work from there.

$$10a^2-5a-15=0 \rightarrow 5(2a^2-a-3)=0$$

To factor the polynomial in the parentheses, I'm going to use the trial and error method since 2 and 3 have so few factors.

$$(2a-3)(a+1) \; or \; (2a+3)(a-1)$$
$$or \; (2a-1)(a+3) \; or \; (2a+1)(a-3)$$

Check the OI terms. In the top case we have 2a-3a=-a and -2a+3a=+a. In the bottom case we have 6a-a=+5a and -6a+a=-5a. The middle term we are after is −a, which is the first set of parentheses in the first row. The factored form we want then is:

$$5(2a^2 - a - 3) = 0 \rightarrow 5(2a - 3)(a + 1) = 0$$

From here, proceed to Step 4 as before.

$$5 = 0 \ \ unsolvable$$
$$2a - 3 = 0 \ \rightarrow 2a = 3 \rightarrow a = \frac{3}{2}$$
$$a + 1 = 0 \rightarrow a = -1$$

By "unsolvable" here we just mean that since this equation cannot be true, it cannot contribute to the solution. It doesn't make the whole problem unsolvable. Our solutions, then, come from the other two factors: $a = \left\{\frac{3}{2}, -1\right\}$

Example 4. $x^2 = 3 + 2x$
We have no parentheses, but we also have terms on both sides of the equation here. Before we can proceed, we need to complete Step 2. Subtract everything on the right side and move it to the left side. Arrange the terms in standard descending order.

$$x^2 = 3 + 2x \rightarrow x^2 - 2x - 3 = 0$$

We can now move on to factoring. This is a relatively easy one. The last constant is negative, and the larger factor must take the negative sign.

$$x^2 - 2x - 3 = 0 \rightarrow (x - 3)(x + 1) = 0$$

In Step 4 we set each factor equal to zero and then solve the resulting equations.

$$x - 3 = 0 \rightarrow x = 3$$
$$x + 1 = 0 \rightarrow x = -1$$

So, our solution set is $x = \{3, -1\}$.

Example 5. $z(2z + 7) = 4$
This problem starts all the way up in Step 1. We don't have anything equal zero, and we have parentheses we will need to simplify before we can do Step 3.

$$z(2z + 7) = 4 \rightarrow 2z^2 + 7z = 4$$

Now in Step 2, move the 4 over so the right side can be zero.

$$2z^2 + 7z = 4 \rightarrow 2z^2 + 7z - 4 = 0$$

Refactor from this point. We can use factoring by grouping. ac=8, and factors of 8 with a difference of 7 are 1x8.

$$2z^2 + 7z - 4 = 0 \rightarrow 2z^2 + 8z - z - 4 = 0 \rightarrow 2z(z+4) - 1(z+4) = 0 \rightarrow (z+4)(2z-1) = 0$$

Since the second pair of terms starts with a negative (-z-4), I factored out the negative 1 giving me the z+4 common factor I needed to proceed. As with previous problems, set each factor equal to zero and solve.

$$z + 4 = 0 \rightarrow z = -4$$
$$2z - 1 = 0 \rightarrow 2z = 1 \rightarrow z = \frac{1}{2}$$

So our solution set is $z = \left\{-4, \frac{1}{2}\right\}$.

Example 6. $x^3 + 3x = x^2 + 3$
It's certainly possible to have a situation where one of your factors can't give you a solution. Start with Step 2 and move everything to one side.

$$x^3 + 3x = x^2 + 3 \rightarrow x^3 - x^2 + 3x - 3 = 0$$

This has 4 terms, so use factor by grouping.

$$x^3 - x^2 + 3x - 3 = 0 \rightarrow x^2(x-1) + 3(x-1) = 0 \rightarrow (x-1)(x^2+3) = 0$$

Setting each factor equal to zero we get:

$$x - 1 = 0 \rightarrow x = 1$$
$$x^2 + 3 = 0 \rightarrow x^2 = -3 \ \ unsolvable$$

The first factor gave us our only solution x=1. But the second one is not solvable. All perfect squares, at least of real numbers, are positive. So there is no real number whose square is -3. This factor can't contribute any solutions (until we learn about complex numbers).

Practice Problems.
Solve each equation for the variable.

1. $(3m-7)(m-2) = 0$
2. $t(5t+6) = 0$
3. $(2x+7)(x^2+2x-3) = 0$
4. $n^2 = 121$
5. $m^3 = 4m$
6. $(q+6)(q-12)(q-1)(q-2) = 0$
7. $x^2 = 24 - 5x$
8. $6r^2 - r - 2 = 0$
9. $(n+7)(n+4) = n + 4$
10. $g(g-7) = -10$
11. $16s^3 - 9s = 0$
12. $(x+1)(6x^2+x-12) = 0$

13. $x^2 + (x+1)^2 = (x+2)^2$
14. $x^2 + (x+7)^2 = (2x-3)^2$
15. $206 = -16t^2 + 180t + 6$