Single Variable Integration Review Key

1) $\int 2x^2 - 1 + x^{-1}\, dx = \frac{2}{3}x^3 - x + \ln x + C$

2) $\int \frac{x^3 - 2x + 7}{x^2}\, dx = \int x - \frac{2}{x} + 7x^{-2}\, dx = \frac{1}{2}x^2 - 2\ln x - \frac{7}{x} + C$

3) $\int 5\sin x + 3\cos x - 2\sec^2 x\, dx = -5\cos x + 3\sin x - 2\tan x + C$

4) $\int e^t + 2t^{-1/3}\, dt = e^t + \cancel{2}\, t^{2/3} \cdot \frac{3}{\cancel{2}} + C = e^t + 3\sqrt[3]{t^2} + C$

5) $\int t^{3/2} - \tan t\, dt = \frac{2}{5}t^{5/2} + \ln|\cos t| + C$

6) $\int s^6 + 3s^4 + 3s^2 + 1\, ds = \frac{1}{7}s^7 + \frac{3}{5}s^5 + s^3 + s + C$

7) $\int \sin^3(q)\cos q\, dq$ $\quad u = \sin q$, $du = \cos q\, dq$ $\quad \int u^3\, du = \frac{u^4}{4} + C = \frac{1}{4}\sin^4 q + C$

8) $\int (x+1)5^{(x+1)^2}\, dx$ $\quad u = (x+1)^2$, $du = 2(x+1)\,dx \Rightarrow \frac{1}{2}du = (x+1)\,dx$

$\int \frac{1}{2}5^u\, du = \frac{1}{2\ln 5}5^u + C = \frac{1}{2\ln 5}5^{(x+1)^2} + C$

9) $\int \frac{2^{1/s}}{s^2}\, ds$ $\quad u = 1/s$, $du = -\frac{1}{s^2}ds \Rightarrow -du = \frac{1}{s^2}ds$ $\quad \int -2^u\, du = -\frac{1}{\ln 2}2^u + C$

$= -\frac{1}{\ln 2}2^{1/s} + C$

10) $\int \frac{1}{\sqrt{1-x}}\, dx$ $\quad u = 1-x$, $-du = +dx$ $\quad \int -u^{-1/2}\, du = -2u^{1/2} + C$

$= -2\sqrt{1-x} + C$

11) $\int y^2\sqrt{y+1}\,dy$ $\quad u=\sqrt{y+1}$

$u^2=y+1$

$u^2-1=y \Rightarrow 2u\,du=dy$

$\int (u^2-1)^2 u\cdot 2u\,du = \int (u^4-2u^2+1)2u^2\,du = \int 2u^6-4u^4+2u^2\,du$

$= \frac{2}{7}u^7-\frac{4}{5}u^5+\frac{2}{3}u^3+C = \frac{2}{7}(y+1)^{7/2}-\frac{4}{5}(y+1)^{5/2}+\frac{2}{3}(y+1)^{3/2}+C$

12) $\int \frac{\frac{1}{2}t\ln(1+t^2)}{1+t^2}\,dt$ $\quad u=\ln(1+t^2)$

$du=\frac{2t}{1+t^2}dt \Rightarrow \frac{1}{2}du=\frac{t}{1+t^2}$

$\int \frac{1}{2}\cdot\frac{1}{2}u\,du = \frac{1}{8}u^2+C = \frac{1}{8}[\ln(1+t^2)]^2+C$

13) $\int \frac{1}{3+25w^2}\,dw$ $\quad u=5w$

$du=5\,dw \Rightarrow \frac{1}{5}du=dw$ $\quad \int \frac{1}{5}\cdot\frac{1}{3+u^2}du =$

$\frac{1}{5}\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{u}{\sqrt{3}}\right)+C = \frac{1}{5\sqrt{3}}\arctan\left(\frac{5w}{\sqrt{3}}\right)+C$

14) $\int \frac{\arccos r}{\sqrt{1-r^2}}\,dr$ $\quad u=\arccos r$

$-du=\frac{-1}{\sqrt{1-r^2}}dr$ $\quad \int -u\,du = -\frac{1}{2}u^2+C$

$= -\frac{1}{2}\arccos^2 r+C$

15) $\int \sqrt{1+\sinh^2 s}\,ds = \int \sqrt{\cosh^2 s}\,ds = \int \cosh s\,ds = \sinh s+C$

16) $\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}dx$ $\quad u=\cosh x$

$du=\sinh x\,dx$ $\quad \int \frac{1}{u}du = \ln|\cosh x|+C$

17) $\int 2e^{-x}\cosh x\,dx = \int \cancel{2}e^{-x}\left(\frac{e^x+e^{-x}}{\cancel{2}}\right)dx = \int 1+e^{-2x}\,dx =$

$x-\frac{1}{2}e^{-2x}+C$

18) $\int (u^2-1)e^u\,du$

	u	dv
$+$	u^2-1	e^u
$-$	$2u$	e^u
$+$	2	e^u
$-$	0	e^u

$$= (u^2-1)e^u - 2ue^u + 2e^u + C = e^u(u^2-2u+1) + C$$

19) $\int q^e + e^q\,dq = \dfrac{q^{e+1}}{e+1} + e^q + C$

20) $\int \arctan(t)\,dt$ $\quad u = \arctan(t) \quad dv = dt$

$\quad du = \dfrac{1}{1+t^2} \quad v = t$

$$= t\arctan t - \int \frac{t}{1+t^2}\,dt = t\arctan t - \frac{1}{2}\ln(1+t^2) + C$$

21) $\int \cos^4(y)\,dy = \int \left[\frac{1}{2}(1+\cos(2y))\right]^2 dy = \frac{1}{4}\int 1 + 2\cos 2y + \cos^2 2y\,dy$

$$= \frac{1}{4}\int 1 + 2\cos 2y + \frac{1}{2}(1+\cos 4y)\,dy = \frac{1}{4}\int \frac{3}{2} + 2\cos 2y + \frac{1}{2}\cos 4y\,dy$$

$u = 2y$ $\quad w = 4y$

$du = 2dy$ $\quad dw = 4dy$

$\frac{1}{2}du = dy$ $\quad \frac{1}{4}dw = dy$

$\int \cos u\,du$ $\quad \int \frac{1}{8}\cos w\,dw$

$\sin u$ $\quad \frac{1}{8}\sin w$

$$\frac{1}{4}\left[\frac{3}{2}y + \sin 2y + \frac{1}{8}\sin 4y\right] + C$$

$$\frac{3}{8}y + \frac{1}{4}\sin 2y + \frac{1}{32}\sin 4y + C$$

22) $\int \cos(3z)\cos(z)\,dz = \int \frac{1}{2}\left[\cos(3z-z) + \cos(3z+z)\right]dz =$

$$\frac{1}{2}\int \cos(2z) + \cos 4z\,dz = \frac{1}{2}\left[\frac{1}{2}\sin 2z + \frac{1}{4}\sin 4z\right] + C =$$

$u = 2z$ $\quad w = 4z$

$du = 2dz$ $\quad dw = 4dz$

$\frac{1}{2}du = dz$ $\quad \frac{1}{4}dw = dz$

$\int \frac{1}{2}\cos u\,du$ $\quad \int \frac{1}{4}\cos w\,dw$

$\frac{1}{2}\sin u$ $\quad \frac{1}{4}\sin w$

$$\frac{1}{4}\sin 2z + \frac{1}{8}\sin 4z + C$$

23) $\int e^x \sin x \, dx$ $\quad u = \sin x \quad dv = e^x dx$ $\quad du = \cos x \, dx \quad v = e^x$

$= e^x \sin x - \int e^x \cos x \, dx$ $\quad u = \cos x \quad dv = e^x$ $\quad du = -\sin x \, dx \quad v = e^x$

$= e^x \sin x - \left[e^x \cos x - \int -e^x \sin x \, dx \right]$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$+ \int e^x \sin x \, dx \qquad\qquad + \int e^x \sin x \, dx$$

$$\frac{2 \int e^x \sin x \, dx}{2} = \frac{e^x \sin x - e^x \cos x}{2}$$

$$\int e^x \sin x \, dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x) + C$$

24) $\int \alpha \sqrt{\alpha + 3} \, d\alpha$

$u = \sqrt{\alpha + 3}$

$u^2 = \alpha + 3$

$u^2 - 3 = \alpha$

$2u \, du = d\alpha$

$\int (u^2 - 3) u \cdot 2u \, du =$

$\int 2u^4 - 6u^2 \, du$

$\frac{2}{5} u^5 - 2u^3 + C$

$\frac{2}{5}(\alpha + 3)^{5/2} - 2(\alpha + 3)^{3/2} + C$

(this problem can also be done by parts)

25) $\int \frac{2x^3 - 5x^2 + 4x - 4}{x^2 - x} \, dx$

$$\begin{array}{r} 2x - 3 \\ x^2 - x \enclose{longdiv}{2x^3 - 5x^2 + 4x - 4} \\ -2x^3 - 2x^2 \\ \hline -3x^2 + 4x \\ +3x^2 + 3x \\ \hline 7x - 4 \end{array}$$

$= \int 2x - 3 + \frac{7x - 4}{x^2 - x} \, dx$

$x^2 - 3x + \int \frac{7x - 4}{x(x-1)} \, dx = x^2 - 3x + \int \frac{A}{x} + \frac{B}{x-1} \, dx = x^2 - 3x + A \ln x + B \ln|x - 1| + C$

$A(x-1) + B(x) = 7x - 4$

$x = 0: \; -A = -4 \quad A = 4 \quad ; \; x = 1: \; B = 3$

$\Rightarrow x^2 - 3x + 4 \ln x + 3 \ln|x - 1| + C$

$= x^2 - 3x + \ln|x^4 (x-1)^3| + C$

26) $\int \frac{t-28}{(t-3)(t+2)}\,dt = \int \frac{A}{t-3} + \frac{B}{t+2}\,dt = A\ln|t-3| + B\ln|t+2| + C$

$A(t+2) + B(t-3) = t-28$

$t=-2: \; -5B = -30 \Rightarrow B = 6$

$t=3: \; 5A = -25 \Rightarrow A = -5$

$= -5\ln|t-3| + 6\ln|t+2| + C$

$= \ln\left|\frac{(t+2)^6}{(t-3)^5}\right| + C$

27) $\int \frac{\sec^2\theta}{\tan\theta(\tan\theta - 1)}\,d\theta$ $\quad u = \tan\theta$, $du = \sec^2\theta\,d\theta$ $\quad \int \frac{du}{u(u-1)} = \int \frac{A}{u} + \frac{B}{u-1}\,du$

$= A\ln u + B\ln|u-1| + C = A\ln|\tan\theta| + B\ln|\tan\theta - 1| + C$

$A(u-1) + Bu = 1$

$u=1: \; B = 1 \;; \quad u=0: \; -A = 1 \quad A = -1$

$= -\ln|\tan\theta| + \ln|\tan\theta - 1| + C =$

$\ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| + C = \ln|1 - \cot\theta| + C$

28) $\int \sqrt{9-4x^2}\,dx$

$2x = 3\sin\theta$

$2dx = 3\cos\theta\,d\theta$

$dx = \frac{3}{2}\cos\theta\,d\theta$

$\sqrt{9 - 9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9\cos^2\theta} = 3\cos\theta$

$\int 3\cos\theta \cdot \frac{3}{2}\cos\theta\,d\theta = \frac{9}{2}\int \cos^2\theta\,d\theta =$

$\frac{9}{2}\cdot\frac{1}{2}\int 1 + \cos 2\theta\,d\theta = \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$

$u = 2\theta$

$du = 2d\theta$

$\frac{1}{2}du = d\theta$

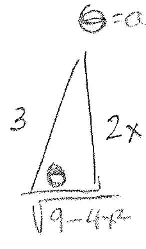
$\int \frac{1}{2}\cos u\,du$

$\frac{1}{2}\sin u$

$\frac{9}{4}\theta + \frac{9}{4}\cdot\frac{1}{\cancel{2}}\cancel{2}\sin\theta\cos\theta + C$

$\frac{2x}{3} = \sin\theta$

$\theta = \arcsin\left(\frac{2x}{3}\right)$



$= \frac{9}{4}\arcsin\left(\frac{2x}{3}\right) + \frac{\cancel{9}}{\cancel{4}_2}\cdot\frac{2x}{\cancel{3}}\cdot\frac{\sqrt{9-4x^2}}{\cancel{3}} + C$

$= \frac{9}{4}\arcsin\left(\frac{2x}{3}\right) + \frac{x}{2}\sqrt{9-4x^2} + C$