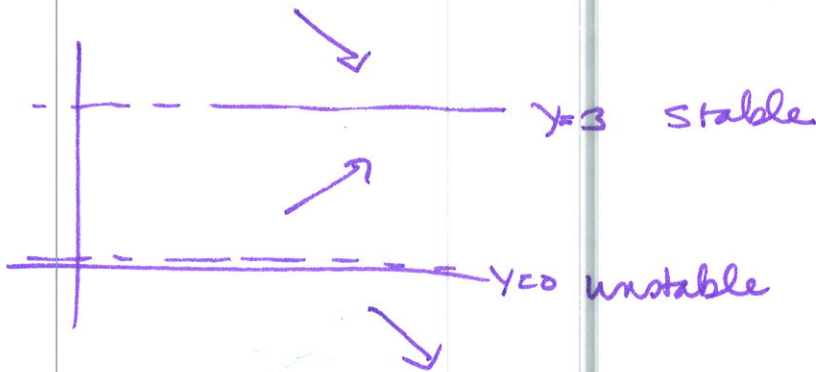


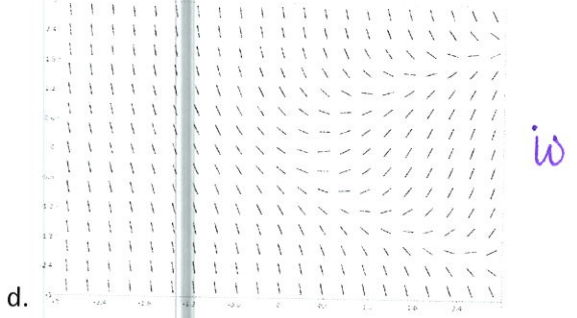
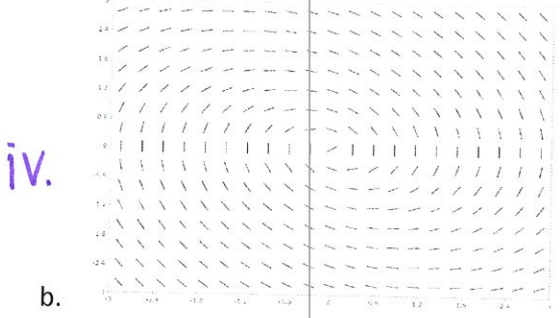
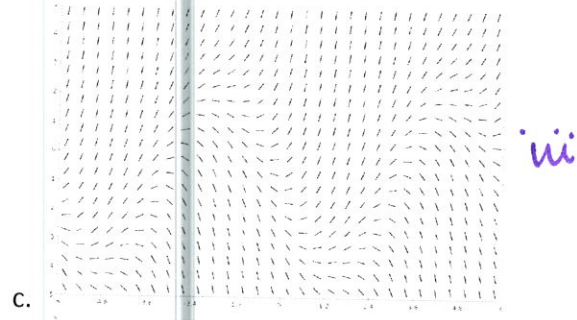
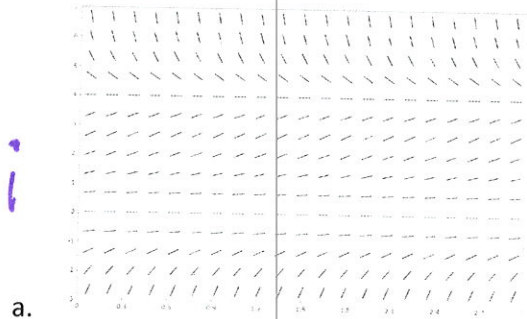
Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Sketch the direction field for $y' = y(3 - y)$. Label any equilibria and classify its stability.



2. Match the direction fields to the corresponding differential equations.

- i. $y' = \frac{1}{4}y^2(4 - y)$ *a*
- ii. $y' = 2t - 1 - y^2$ *d*
- iii. $y' = 3 \sin(t) + 1 + y$ *c*
- iv. $y' = -\frac{2t+y}{2y}$ *b*



3. Classify the following differential equations by i) linearity, ii) order, iii) ordinary or partial.

a. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

linear, 4th order, partial

b. $u_t + uu_x = 1 + u_{xx}$

nonlinear, 2nd order, partial

c. $y''' - 3y'' + 2y' = 0$

linear, 3rd order, ordinary

d. $2t^2y'' + 3ty' - y = 0$

linear, 2nd order, ordinary

e. $\frac{d^3y}{dt^3} + t\frac{dy}{dt} + (\cos^2 y)t = t^3$

nonlinear, 3rd order, ordinary

4. Solve $2y' - y = e^{\frac{t}{3}}$ by the method of integrating factors.

$$y' - \frac{1}{2}y = \frac{1}{2}e^{\frac{t}{3}}$$

$$e^{-\frac{1}{2}t} y - \frac{1}{2}e^{-\frac{1}{2}t} y = \frac{1}{2}e^{-\frac{1}{6}t}$$

$$\int (e^{-\frac{1}{2}t} y)' = \int \frac{1}{2}e^{-\frac{1}{6}t} dt$$

$$e^{-\frac{1}{2}t} y = \frac{1}{2}(-6)e^{-\frac{1}{6}t} + C$$

$$y = -3e^{\frac{t}{3}} + Ce^{\frac{1}{2}t}$$

$$\mu = e^{\int -\frac{1}{2} dt} = e^{-\frac{1}{2}t}$$

5. Solve $y' + y^2 \sin x = 0$ by separation of variables.

$$\frac{dy}{dx} = -y^2 \sin x$$

$$\int -\frac{dy}{y^2} = \int \sin x dx$$

$$\frac{1}{y} = -\cos x + C$$

$$y(x) = \frac{1}{C - \cos x}$$

6. Find the general solution for $y' = -\frac{4x+3y}{2x+y}$

$$y' = v'x + v = -\frac{4x+3vx}{2x+vx} = -\frac{x(4+3v)}{x(2+v)}$$

$$y = vx \\ y' = v'x + v$$

$$v'x = \frac{-3v-4}{2+v} - v \frac{(2+v)}{(2+v)} = \frac{-3v-4-2v-v^2}{2+v} = \frac{-5v-4-v^2}{2+v}$$

$$\frac{2+v}{v^2+5v+4} dv = -\frac{1}{x} dx \Rightarrow \frac{2+v}{(v+1)(v+4)} dv = -\frac{1}{x} dx \Rightarrow \int \frac{-1}{v+1} + \frac{2}{v+4} dv = \int -\frac{1}{x} dx$$

$$\frac{A}{v+1} + \frac{B}{v+4} = \frac{Av+4A+Bv+B}{(v+1)(v+4)}$$

$$-\ln|v+1| + 2\ln|v+4| = -\ln|x| + C$$

$$\ln\left|\frac{(v+4)^2}{v+1}\right| = \ln\left(\frac{A}{x}\right) \Rightarrow \left|\frac{\left[\frac{y}{x}+4\right]^2}{\frac{y}{x}+1}\right| = \frac{A}{x}$$

$$A+B=1 \\ 4A+B=2$$

$$A=-1 \\ B=2$$

7. A tank has pure water flowing into it at 10 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 L/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 L of water. How much salt is in the tank after 30 minutes?

$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate out}$$

$$\text{Rate in} = \frac{10L}{\text{min}} \cdot \frac{0 \text{ kg}}{\text{min}} \Rightarrow 0 \text{ kg/min}$$

$$\text{Rate out} = \frac{10L}{\text{min}} \cdot \frac{Q}{100L} = \frac{Q}{10} \quad Q(0) = 10$$

$$\frac{dQ}{dt} = 0.1 - \frac{Q}{10} \Rightarrow Q' + \frac{1}{10}Q = 0.1 \quad \mu = e^{\int \frac{1}{10} dt} = e^{\frac{1}{10}t}$$

$$e^{\frac{1}{10}t} Q' + \frac{1}{10} e^{\frac{1}{10}t} Q = 0.1 e^{\frac{1}{10}t}$$

$$\int (e^{\frac{1}{10}t} Q)' = \int 0.1 e^{\frac{1}{10}t}$$

$$e^{\frac{1}{10}t} Q = e^{\frac{1}{10}t} + C$$

$$Q = 1 + C e^{-\frac{1}{10}t}$$

$$Q = 1 + 9 e^{-\frac{1}{10}t}$$

$$C=9$$

$$Q(30) = 1 + 9 e^{-\frac{1}{10} \cdot 30}$$

$$= \boxed{145 \text{ kg}}$$

8. State the intervals over which the ODE $(4-t^2)y' + 2ty = 3t^2$ is guaranteed to have a solution.

$$y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$$

$$t \neq 2, -2$$

$$\text{defined on } (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$$

9. For the nonlinear ODE $y' = \frac{t-y}{1-t^2-y^2}$, sketch the region(s) in the plane where a solution is guaranteed to exist. Be sure to explicitly check all conditions required for these conditions to hold.

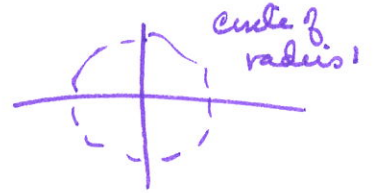
$$f = \frac{t-y}{1-t^2-y^2}$$

$$\frac{\partial f}{\partial y} = \frac{-1(1-t^2-y^2) - (t-y)(-2y)}{(1-t^2-y^2)^2} =$$

still not defined
on same region

$$1-t^2-y^2=0$$

$$t^2+y^2=1$$



defined everywhere
but on circle

10. Rewrite the Bernoulli equation $y' + 2ty + ty^4 = 0$ as a first order linear equation. (You do not need to solve it.)

$$-3y^4 y' - 6ty^3 = 3t$$

$$\boxed{z' - 6tz = 3t}$$

$$(1-n)y^{-n} = -3y^{-4}$$

$$z = y^{-3}$$

$$z' = -3y^{-4} y'$$

11. Describe the behavior of a logistic solution to a population model. Sketch a graph of the direction field or the solution to illustrate.

logistic increases slowly at first then near exponentially then turns over and increases slowly as it approaches a carrying capacity (upper limit)

