MAT 212, Exam #3, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Rewrite $x^2 - 2$ as power series centered at $x_0 = 1$. (4 points)

$(x-1)^2$ x^2-2x+1
$2(x-1)$ $2x-2$
-1 -1
x^2 -2

$$\boxed{(x-1)^2 + 2(x-1) - 1}$$

or taylor

$(x^2-2)(1) = (1^2-2) = -1$
$(x^2-2)'(1) = (2x)(1) = 2(1) = 2$
$(x^2-2)''(1) = 2(1) = 2$
$-1 + \frac{2}{1!}(x-1) + \frac{2}{2!}(x-1)^2$

2. Find the first and second derivatives of $y = \sum_{n=0}^{\infty} a_n (x-2)^n$. (6 points)

$$y' = \sum_{n=1}^{\infty} n a_n (x-2)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-2)^{n-2}$$

3. Rewrite each series so that the power term is x^n. (4 points each)

a. $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$

$$\sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n$$

b. $\sum_{n=0}^{\infty} a_n x^{n+2}$

$$\sum_{n=2}^{\infty} a_{n-2} x^n$$

4. Rewrite $2x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3 \sum_{n=0}^{\infty} a_n x^n$ as a single sum involving x^n. (6 points)

$$\sum_{n=2}^{\infty} 2 n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} 3 a_n x^n$$

$$\sum_{n=1}^{\infty} 2(n+1) n a_{n+1} x^n + \sum_{n=0}^{\infty} 3 a_n x^n$$

$$3a_0 + \sum_{n=1}^{\infty} 3 a_n x^n$$

$$3a_0 + \sum_{n=1}^{\infty} \left[2n(n+1) a_{n+1} + 3a_n\right] x^n$$

5. Use power series to solve $y'' + xy' + 2y = 0$ centered at $x_0 = 0$. Write out at least 4 terms of each solution. (12 points)

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} 2a_n x^n = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + n a_n + 2a_n\right] x^n = 0$$

$$2a_2 + 2a_0 = 0$$
$$a_2 = -a_0$$

$$(n+2)(n+1)a_{n+2} + (n+2)a_n = 0$$

$$a_{n+2} = \frac{-\cancel{(n+2)}a_n}{\cancel{(n+2)}(n+1)} = \frac{-a_n}{n+1}$$

$n=1$ $\quad a_3 = \frac{-a_1}{2} = -\frac{1}{2}a_1$

$n=2$ $\quad a_4 = \frac{-a_2}{3} = -\frac{1}{3}(-a_0) = \frac{1}{3}a_0$

$n=3$ $\quad a_5 = \frac{-a_3}{4} = -\frac{1}{4}\left(-\frac{1}{2}a_1\right) = \frac{1}{8}a_1$

$n=4$ $\quad a_6 = \frac{-a_4}{5} = -\frac{1}{5}\left(\frac{1}{3}a_0\right) = -\frac{1}{15}a_0$

$n=5$ $\quad a_7 = \frac{-a_5}{6} = -\frac{1}{6}\left(\frac{1}{8}\right)a_1 = \frac{-1}{48}a_1$

$n=6$ $\quad a_8 = \frac{-a_6}{7} = -\frac{1}{7}\left(\frac{-1}{15}\right)a_0 = \frac{1}{105}a_0$

$n=7$ $\quad a_9 = \frac{-a_7}{8} = -\frac{1}{8}\left(\frac{-1}{48}a_1\right) = \frac{+1}{384}a_1$

$$y(x) = a_0\left(1 - x^2 + \frac{1}{3}x^4 - \frac{1}{15}x^6 + \frac{1}{105}x^8 + \dots\right) + a_1\left(x - \frac{1}{2}x^3 + \frac{1}{8}x^5 - \frac{1}{48}x^7 + \frac{1}{384}x^9 + \dots\right)$$

6. Rewrite the equation $x^2y'' + (x+1)y' + 3y = 0$ so that it can be solved with a series centered at $x_0 = 2$. Proceed to solve the system only to the point where the equation can be written in a single summation. (8 points)

$X^2 = (x-2)^2 = x^2-4x+4$

$4(x-2) \qquad 4x-8$

$4 \qquad +4$

x^2

$[(x-2)^2+4(x-2)+4]y'' + [(x-2)+3]y' + 3y = 0$

$\sum_{n=2}^{\infty} n(n+1)a_n(x-2)^{n-2} \qquad \sum_{n=1}^{\infty} na_n(x-2)^{n-1} \qquad \sum_{n=0}^{\infty} a_n(x-2)^n$

$x+1 = x-2$

$+3$

$\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^n + \sum_{n=2}^{\infty} 4n(n-1)a_n(x-2)^{n-1} + \sum_{n=2}^{\infty} 4n(n-1)a_n(x-2)^{n-2}$

$+ \sum_{n=1}^{\infty} na_n(x-2)^n + \sum_{n=1}^{\infty} 3na_n(x-2)^{n-1} + \sum_{n=0}^{\infty} 3a_n(x-2)^n = 0$

$\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^n + \sum_{n=1}^{\infty} 4(n+1)na_{n+1}(x-2)^n + \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}(x-2)^n + \sum_{n=0}^{\infty} 3(n+1)a_{n+1}(x-2)^n + \sum_{n=1}^{\infty} na_n(x-2)^n$

$+ \sum_{n=0}^{\infty} 3a_n(x-2)^n = 0$

$$\boxed{\begin{gathered}4(2)(1)a_2(x-2) + 4(2)(1)a_2(1) + 4(3)(2)a_3(x-2) + 3(1)a_1 + 3(2)a_2(x-2) + 1a_1(x-2) + 3a_0 + 3a_1(x-2) + \\ \sum_{n=2}^{\infty}\left[n(n-1)a_n + 4(n+1)na_{n+1} + 4(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + na_n + 3a_n\right](x-2)^n = 0\end{gathered}}$$

7. For each equation, determine the location of any singular points, and for each one, classify it as regular or irregular. (4 points each)

a. $x^2(1-x)^2y'' + 2xy' + 4y = 0$

$\frac{2x}{x^2(1-x)^2} = \frac{2}{x(1-x)^2} \qquad \frac{4}{x^2(1-x)^2}$

$x=0$ regular $\qquad x=1$ irregular

b. $x(x-3)y'' + (x+1)y' - 2y = 0$

$\frac{x+1}{x(x-3)} \qquad \frac{2}{x(x-3)}$

$x=0, x=3$ both are regular

c. $(\sin x)y'' + xy' + 4y = 0$

$\frac{x}{\sin x} \qquad \frac{4}{\sin x} \qquad \lim_{x\to 0}\frac{x}{\sin x} = 1 \qquad x=0$ regular $\lim_{x\to 0}\frac{x^2}{\sin x} = 0$

$x=0$ regular $\qquad x \lim_{x\to n\pi}\frac{(x-n\pi)}{\sin x} = \frac{1\cdot x}{\cos x} = 1(n\pi)$

$\lim \frac{(x-n\pi)^2}{} = \frac{2(x-n\pi)}{\cos x} = 0$