MAT 212, Quiz #11, Spring 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Determine if $y_1 = \sin x, y_2 = \cos x$ are orthogonal under the inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$.

$\int_{-\pi}^{\pi} \sin x \cos x\, dx = \frac{1}{2}\sin^2 x \Big|_{-\pi}^{\pi} = 0$

Yes the are orthogonal since their inner product is zero

2. Express each piecewise function in terms of the unit step function.

a. $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 1, & t \ge 2 \end{cases}$

$(t-2)^2 + 4(t-2)$

$t^2 =$

$t^2 - 4t + 4 + 4t - 4$

$f(t) = t^2 + 4t$

$1 - t^2$

$1 - (t-2)^2 - 4(t-2)$

$f_2(t) = 1 - t^2 - 4t$

$t^2 + u(t-2)(1-t^2) = f(t)$

MATLAB notation

$f(t) = t^2 + u_2(t)[1-(t-2)^2 - 4(t-2)]$ (textbook)

b. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ t-1, & 1 \le t < 2 \\ t-2, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

$t - 1 - t = -1$

$t - 2 - t = -2$

$-t = -(t-3) - 3$

$t - u_1(-1) - (-2)u_2 - u_3 f(t-c)$

$f(t) = t + u_1 + 2u_2 - u_3[-(t-3)-3]$

textbook

$f(t) = t + u(t-1)(t-1-t) +$
$u(t-2)(t-2-t+1) + u(t-3)(2-t)$

Matlab not.

3. Find the inverse Laplace transform (using the table) of each function.

a. $F(s) = \frac{e^{-2s}}{s^2+s-2} = e^{-2s}\left(\frac{1}{(s+2)(s-1)}\right)$

$u_2(t)\left(-\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}\right)$

$f(t) = \frac{1}{3}u_2(t)\left[e^t - e^{-2t}\right]$

$\frac{A}{s+2} + \frac{B}{s-1} =$

$As - A + Bs + 2B = 1$

$A + B = 0 \qquad A = -1/3$

$-A + 2B = 1 \qquad B = 1/3$

$-\frac{1}{3}\left(\frac{1}{s+2}\right) + \frac{1}{3}\left(\frac{1}{s-1}\right)$

b. $F(s) = \frac{s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$

$As^2 + 4A + Bs^2 + Bs + Cs + C = s$

$A + B = 0 \qquad A = -1/5$

$B + C = 1 \qquad B = 1/5$

$4A + C = 0 \qquad C = 4/5$

$-\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s+4}{s^2+4}\right) =$

$-\frac{1}{5}\left(\frac{1}{s+1}\right) + \frac{1}{5}\left(\frac{s}{s^2+4}\right) + \frac{4}{5}\left(\frac{1}{s^2+4}\right)$

$f(t) = -\frac{1}{5}e^{-t} + \frac{1}{5}\cos 2t + \frac{4}{5}\sin 2t$

or $\int_0^t e^{-1(t-\tau)}\cos 2\tau\, d\tau = \int_0^t e^{-\tau}\cos(2(t-\tau))\, d\tau$

4. Use the table to find the Laplace transform of $f(t) = \int_0^t (t-\tau)\cos 2\tau \, d\tau$.

$\int_0^t f(t\ \tau)\, g(\tau) d\tau$

$f(t-\tau) = t \quad g(t) = \cos 2t$

$F(s) = \frac{1}{s^2} \quad G(s) = \frac{s}{s^2+4}$

$\frac{s}{s^2(s^2+4)} = \boxed{\frac{1}{s(s^2+4)}}$

5. Use Laplace transforms to solve.

a. $y'' + 3y' + 2y = u_2(t), y(0) = 0, y'(0) = 1$

$s^2F(s) - s(0) - 1 + 3(sF(s) - 0) + 2F(s) = \frac{e^{-2s}}{s}$

$F(s)(s^2+3s+2) \cancel{-1} = \frac{e^{-2s}}{s} + \frac{s}{s}$

$F(s) = \left(\frac{e^{-2s}+s}{s(s^2+3s+2)}\right) = \frac{e^{-2s}}{s(s+2)(s-1)} + \frac{\cancel{s}\,1}{\cancel{s}(s+2)(s-1)}$

$(s+2)(s-1)$

$\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$ $\quad A+B+C=0$, $3A-B+2C=0$, $2A=1$

$A(s^2+3s+2)+B(s^2-s)+C(s^2+2s)$ $\quad A=\frac{1}{2}\ B=\frac{1}{6}\ C=-\frac{2}{3}$

$\frac{A}{s+2} + \frac{B}{s-1}$ $\quad A+B=0$, $-A+2B=1$ $\quad A=-\frac{1}{3}\ B=\frac{1}{3}$

$As-A+Bs+2B=1$

$\longrightarrow$

b. $y'' + 2y' + 2y = \delta(t-\pi), y(0) = 1, y'(0) = 0$

$s^2F(s) - s + 2(sF(s) - 1) + 2F(s) = e^{-\pi s}$

$F(s)(s^2+2s+2) - s - 2 = e^{-\pi s}$

$F(s)(s^2+2s+2) = e^{-\pi s} + s + 2$

$F(s) = \frac{e^{-\pi s} + s + 2}{s^2+2s+2} = \frac{e^{-\pi s} + (s+1) + 1}{(s+1)^2+1} = \frac{e^{-\pi s}}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$

$y(t) = e^{-t}\cos t + e^{-t}\sin t + u_\pi(t)\sin(t-\pi)e^{-t+\pi}$

5 cont'd

$$e^{-2s}\left(\frac{1/2}{s} + \frac{1/6}{s+2} + \frac{-2/3}{s-1}\right) + -\frac{1}{3}\left(\frac{1}{s+2}\right) + \frac{1}{3}\left(\frac{1}{s-1}\right)$$

$$y(t) = \frac{1}{2}u_2 + \frac{1}{6}e^{-2t}u_2 - \frac{2}{3}u_2 e^{t} - \frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

$$\left(\frac{1}{2} + \frac{1}{e}e^{-2t} - \frac{2}{3}e^{t}\right)u_2 - \frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}$$

Laplace transforms – Table			
$f(t) = L^{-1}\{F(s)\}$	$F(s)$	$f(t) = L^{-1}\{F(s)\}$	$F(s)$
$a \quad t \geq 0$	$\frac{a}{s} \quad s > 0$	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$at \quad t \geq 0$	$\frac{a}{s^2}$	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
e^{-at}	$\frac{1}{s + a}$	$\sin(\omega t + \theta)$	$\frac{s \sin \theta + \omega \cos \theta}{s^2 + \omega^2}$
te^{-at}	$\frac{1}{(s + a)^2}$	$\cos(\omega t + \theta)$	$\frac{s \cos \theta - \omega \sin \theta}{s^2 + \omega^2}$
$\frac{1}{2} t^2 e^{-at}$	$\frac{1}{(s + a)^3}$	$t \sin \omega t$	$\frac{2\omega s}{(s^2 + \omega^2)^2}$
$\frac{1}{(n-1)!} t^{n-1} e^{-at}$	$\frac{1}{(s + a)^n}$	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
e^{at}	$\frac{1}{s - a} \quad s > a$	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
te^{at}	$\frac{1}{(s - a)^2}$	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2} \quad s > \lvert\omega\rvert$
$\frac{1}{b - a}(e^{-at} - e^{-bt})$	$\frac{1}{(s + a)(s + b)}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
$\frac{1}{a^2}[1 - e^{-at}(1 + at)]$	$\frac{1}{s(s + a)^2}$	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$
t^n	$\frac{n!}{s^{n+1}} \quad n = 1,2,3..$	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$
$t^n e^{at}$	$\frac{n!}{(s - a)^{n+1}} \quad s > a$	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s + a)^{n+1}} \quad s > a$	$1 - e^{-at}$	$\frac{a}{s(s + a)}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s + a)}$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}} \quad s > 0$	$f(t - t_1)$	$e^{-t_1 s} F(s)$
$g(t) \cdot p(t)$	$G(s) \cdot P(s)$	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
$\int f(t)dt$	$\frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$	$\delta(t)$ *unit impulse*	$1 \quad$ *all s*
$\frac{df}{dt}$	$sF(s) - f(0)$	$\frac{d^2 f}{df^2}$	$s^2 F(s) - sf(0) - f'(0)$
$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{n-1}(0)$		