

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Use the definition of the Laplace transform $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$ to find $\mathcal{L}\{1 + \cosh 5t\}$ (8 points)

$$\int_0^{\infty} e^{-st} (1 + \cosh 5t) dt = \int_0^{\infty} e^{-st} \left(\frac{e^{-5t} + e^{5t}}{2} \right) dt = \int_0^{\infty} e^{-st} \left(\frac{1}{2} e^{-t(s+5)} + \frac{1}{2} e^{-t(s-5)} \right) dt$$

$$= \left. \left(\frac{1}{3} e^{-3t} + \frac{1}{2} \cdot \frac{-1}{s+5} e^{-t(s+5)} + \frac{1}{2} \cdot \frac{-1}{s-5} e^{-t(s-5)} \right) \right|_0^{\infty} =$$

$$0 + \frac{1}{3} - 0 + \frac{1}{2(s+5)} - 0 + \frac{1}{2(s-5)} = \frac{1}{3} + \frac{1}{2} \left[\frac{s-5 + s+5}{s^2 - 25} \right] =$$

$$\frac{1}{3} + \frac{s}{s^2 - 25}$$

2. Use the table of Laplace transforms to find Laplace transforms or inverse Laplace transforms as indicated. (3 points each)

a. $\mathcal{L}\{(1+t)^2\} = \mathcal{L}\{1+2t+t^2\} =$

$$\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}$$

b. $\mathcal{L}\{te^t\}$

$$= -\frac{d}{ds} \left[\frac{1}{s-1} \right] = \frac{+1}{(s-1)^2}$$

a. $\mathcal{L}\{e^{-2t} \sin 3\pi t\}$

$$\frac{3\pi}{(s+2)^2 + 9\pi^2}$$

$$d. \mathcal{L}\{\sin t * t^2 e^{2t}\} = \mathcal{L}\left\{\int_0^t \sin(t-\tau) \cdot \tau^2 e^{2\tau} d\tau\right\}$$

$$\frac{1}{s^2+1} \cdot \frac{d}{ds} \left\{ \frac{d}{ds} \left[\frac{1}{s-2} \right] \right\} =$$

$$\frac{1}{s^2+1} \cdot \frac{2}{(s-2)^3}$$

$$e. \mathcal{L}\left\{\frac{\sin t}{t}\right\}$$

$$\arctan\left(\frac{1}{s}\right)$$

$$f. f(t) = \begin{cases} \cos \pi t, & 0 \leq t < 2 \\ t & t \geq 2 \end{cases}, \mathcal{L}\{f(t)\}$$

$$\cos \pi t - \left[(t - \cos \pi t) \right] u(t-2)$$

$$\frac{s}{s^2+\pi^2} - \left[\frac{1}{s} + \frac{2}{s} - \frac{s \cos 2 - \pi \sin 2}{s^2+\pi^2} \right] e^{-2s}$$

$$g. \mathcal{L}\left\{\frac{1}{2} \int_0^t (t-\tau)^3 \sin 2\tau d\tau\right\}$$

$$\frac{1}{2} \cdot \frac{6}{s^4} \cdot \frac{2}{s^2+4} = \frac{6}{s^4(s^2+4)}$$

$$h. \mathcal{L}\{\delta(t-1)\}$$

$$e^{-s}$$

$$i. \mathcal{L}^{-1}\left\{\frac{1}{2} - \frac{2}{s^5}\right\}$$

$$\frac{1}{2} \delta(t) - \frac{1}{12} t^4$$

$$j. \mathcal{L}^{-1}\left\{\frac{9-17s}{s^2+81}\right\} = \frac{9}{s^2+81} - \frac{17s}{s^2+81}$$

$$\sin 9t - 17 \cos 9t$$

$$k. \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{s} \cdot \frac{1}{s^2+4}$$

$$\frac{1}{2} \int_0^t \sin 2\tau \, d\tau$$

$$l. \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \frac{1}{s^2} \cdot \frac{1}{s^2-1}$$

$$\int_0^t (t-\tau) \sinh \tau \, d\tau$$

$$m. \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = e^{-\pi s} \cdot \frac{1}{s^2+1}$$

$$\sin(t-\pi) u(t-\pi)$$

3. Write the function $f(t) = \begin{cases} t^2, & 0 \leq t < 1 \\ t^3, & 1 \leq t < 2 \\ t, & t \geq 2 \end{cases}$ in terms of the unit step function. (6 points)

$$t^2 + (t^3 - t^2)u(t-1) + (t - t^3)u(t-2)$$

4. Use Laplace transforms to solve the IVP $y'' + 4y' + 8y = e^{-t}, y(0) = 0, y'(0) = 1$. (10 points)

$$s^2 Y(s) - s(0) - 1 + 4sY(s) - 0 + 8Y(s) = \frac{1}{s+1}$$

$$Y(s)(s^2 + 4s + 8) = \frac{1}{s+1} + 1$$

$$Y(s) = \frac{1}{(s+1)(s^2+4s+8)} + \frac{s+1}{(s+1)(s^2+4s+8)} = \frac{s+2}{(s+1)(s^2+4s+8)}$$

$$\frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8} = \frac{As^2+4As+8A+Bs^2+Bs+C}{(s+1)(s^2+4s+8)}$$

$$\begin{aligned} A+B &= 0 \\ 4A+B+C &= 1 \\ 8A+C &= 2 \end{aligned}$$

5. Write $x^4 + 3x$ as a power series in terms of $x+1$. (6 points)

$f(x) = x^4 + 3x$	$f(1) =$	$-2 - (x+1) + \frac{12(x+1)^2}{2!} - \frac{24(x+1)^3}{3!} + \frac{24(x+1)^4}{4!}$
$f'(x) = 4x^3 + 3$	$f'(1) = -1$	
$f''(x) = 12x^2$	$f''(1) = 12$	$-2 - (x+1) + 6(x+1)^2 - 4(x+1)^3 + (x+1)^4$
$f'''(x) = 24x$	$f'''(1) = 24$	
$f^{(4)}(x) = 24$	$f^{(4)}(1) = 24$	
$f^{(5)}(x) = 0$	$f^{(5)}(1) = 0$	

6. Rewrite the power series $\sum_{n=3}^{\infty} a_n x^{n-2}$ so that the index starts at $n=0$. (4 points)

$$\sum_{n=0}^{\infty} a_{n+3} x^{n+1}$$

7. Write $\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x \sum_{n=1}^{\infty} 4a_n x^{n-1}$ as a single sum with x^n . (5 points)

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 4a_n x^n$$

$$a_2(2)(1)x^0 + \sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} 4a_n x^n$$

$$2a_2 + \sum_{n=1}^{\infty} [a_{n+2}(n+2)(n+1) + 4a_n] x^n = 0$$

8. For each equation, identify any singular points and classify them as regular or irregular. (5 points each)

a. $\frac{x^3-1}{x^2+1}y'' + (x^2-1)y' + y = x$

$$y'' + \frac{(x^2-1)(x^2+1)}{x^3-1} y' + \frac{x^2+1}{x^3-1} y = \frac{x(x^2+1)}{x^3-1}$$

regular at $x=1$

$$y'' + \frac{(x+1)(x^2+1)}{x^2+x+1} y' + \frac{(x^2+1)}{(x-1)(x^2+x+1)} y = \frac{x(x^2+1)}{x^3-1}$$

never 0

$$\lim_{x \rightarrow 1} \frac{(x^2+1)(x-1)^k}{(x-1)(x^2+x+1)} = \text{defined} \rightarrow 0 \text{ at } x=1$$

Singular at $x=1$

b. $(x^2 + 6x)y'' + (3x + 9)y' - 3y = 0$

$$y'' + \frac{3(x+3)}{x(x+6)} y' - \frac{3y}{x(x+6)} = 0$$

singular at $x=0, x=-6$

$$\lim_{x \rightarrow 0} \frac{3(x+3)}{x(x+6)} \cdot x = \text{defined} \quad \lim_{x \rightarrow 0} \frac{3(x+3)}{x(x+6)} x^2 = \text{defined}$$

$$\lim_{x \rightarrow -6} \frac{3(x+3)(x+6)}{x(x+6)} = \text{defined} \quad \lim_{x \rightarrow -6} \frac{3(x+3)(x+6)^k}{x(x+6)} = \text{defined}$$

both are regular

c. $xy'' + (\sin x)y' + xy = 0$

$$y'' + \frac{\sin x}{x} y' + y = 0$$

$x=0$ singular

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot x = 0 \text{ defined}$$

regular