

**Instructions:** Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Estimate the solution of the ODE  $\frac{dy}{dx} = y \cos x$ ,  $y(0) = 1$  using  $\Delta t = 0.1$  using two complete steps of Runge-Kutta. (12 points)

$$x_0 = 0 \quad y_0 = 1 \quad k_{01} = (1) \cos(0) = 1 \quad k_{02} = (1 + .05(1)) \cos(.05) = 1.049$$

$$k_{03} = (1 + .05(1.049)) \cos(.05) = 1.0511$$

$$k_{04} = (1 + .1(1.0511)) \cos(.1) = 1.09959$$

$$x_1 = 0.1 \quad y_1 = \frac{1}{6} (1 + 2(1.049) + 2(1.0511) + 1.09959) \cdot 0.1 + 1 = 1.1049965$$

$$k_{11} = 1.105 \cos(.1) = 1.099476$$

$$k_{12} = (1.099476 \cdot .05 + 1.105) \cos(.15) = 1.1469$$

$$k_{13} = (1.1469 \cdot .05 + 1.105) \cos(.15) = 1.14929$$

$$k_{14} = (1.14929 \cdot .05 + 1.105) \cos(.2) = 1.1956$$

$$x_2 = 0.2 \quad y_2 = \frac{1}{6} (.1) (1.1956 + 2(1.14929) + 2(1.1469) + 1.099476) + 1.105$$

$$= 1.21979$$

2. Three tanks (A, B, and C) are coupled together. Suppose that tank A has 500L of water containing 10 kg of salt, that tank B has 400L of pure water, and that tank C has 1000L of water with 100kg of salt. Pure water flows into tank A at a rate of 5L/s; the well-mixed solution flows into tank B at the same rate, and tank B flows into tank C at the same rate. The well-mixed solution flows out of tank C at the same rate. Write the system that models the amount of salt in each tank at time  $t$ . [You don't need to solve, just set it up.] (10 points)

$$\frac{dA}{dt} = -\frac{A}{500} \cdot 5$$

$$\frac{dB}{dt} = \frac{A}{100} - \frac{B}{400} \cdot 5$$

$$\frac{dC}{dt} = \frac{B}{80} - \frac{C}{1000} \cdot 5$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix}' = \begin{bmatrix} -\frac{1}{100} & 0 & 0 \\ \frac{1}{100} & -\frac{1}{80} & 0 \\ 0 & \frac{1}{80} & -\frac{1}{200} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

$$A(0) = 10$$

$$B(0) = 0$$

$$C(0) = 100$$

3. Write the system  $\begin{cases} x_1' = x_2 \\ x_2' = 2x_3 \\ x_3' = 3x_4 \\ x_4' = 4x_1 \end{cases}$  as a single fourth-order equation. (8 points)

$$\begin{aligned} x_2 &= x_1' \\ x_2' &= x_1'' \\ x_2' &= 2x_3 \\ \frac{1}{2}x_1'' &= x_3 \\ \frac{1}{2}x_1''' &= x_3' \\ \frac{1}{2}x_1''' &= 3x_4 \\ \frac{1}{6}x_1''' &= x_4 \\ \frac{1}{6}x_1^{IV} &= x_4' \end{aligned}$$

$$\begin{aligned} x_4' &= 4x_1 \\ \frac{1}{6}x_1^{IV} &= 4x_1 \\ x_1^{IV} &= 24x_1 \\ \boxed{x_1^{IV} - 24x_1 &= 0} \end{aligned}$$

4. Rewrite  $y^{IV} + 2y''' + y' + y = \cos 2t - 6 \sin 2t$  as a system of first order equations. (You don't need to solve.) (8 points)

$$\begin{aligned} y &= x_1 \\ y' &= x_2 = x_1' \\ y'' &= x_3 = x_2' \\ y''' &= x_4 = x_3' \\ y^{IV} &= x_4' \end{aligned}$$

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= -2x_4 - x_2 - x_1 \end{aligned}$$

$$\vec{X}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & 0 & -2 \end{bmatrix} \vec{X} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos 2t - 6 \sin 2t \end{bmatrix}$$

5. Set up the spring problem shown below as a system of equations. (You don't need to solve.) Let  $m_1 = 3, m_2 = 1, k = 5$ . (10 points)

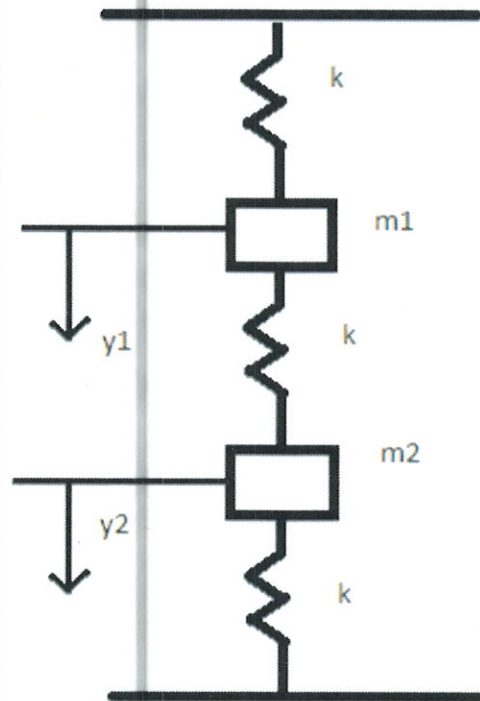
$$m_1 y_1'' = -2ky_1 - k y_2$$

$$m_2 y_2'' = -2ky_2 - ky_1$$

$$3y_1'' = -10y_1 - 5y_2$$

$$y_2'' = -5y_1 - 10y_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'' = \begin{bmatrix} -10/3 & -5/3 \\ -5 & -10 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



6. Solve  $\vec{x}' = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix} \vec{x}$ . Write the general solution (with real terms only). Plot several sample trajectories. (10 points)

$$(1-\lambda)(-2-\lambda) + 3 = 0$$

$$\lambda^2 - \lambda + 2\lambda - 2 + 3 = 0$$

$$\lambda^2 + \lambda + 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\lambda = \frac{-1 \pm \sqrt{3}i}{2}$$

$\lambda_1 =$

$$\begin{bmatrix} 1 - \frac{-1 + \sqrt{3}i}{2} & 3 \\ -1 & -2 - \frac{-1 + \sqrt{3}i}{2} \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 3 - \sqrt{3}i \\ 2 \end{bmatrix}$$

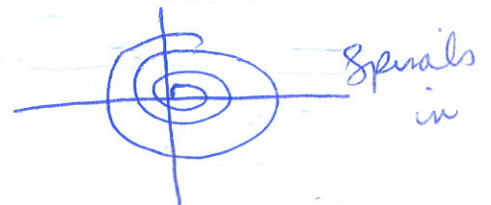
$$\vec{v}_2 = \begin{bmatrix} -3 + \sqrt{3}i \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{3}i}{2} & 3 \\ -1 & -\frac{3}{2} - \frac{\sqrt{3}i}{2} \end{bmatrix}$$

$$x_1 = \left(-\frac{3}{2} - \frac{\sqrt{3}i}{2}\right) x_2$$

$$x_2 = x_2$$

$$\vec{x} = c_1 e^{-\frac{\sqrt{3}}{2}t} \begin{pmatrix} -3 \cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ 2 \cos\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix} + c_2 e^{-\frac{\sqrt{3}}{2}t} \begin{pmatrix} -3 \sin\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3} \cos\left(\frac{\sqrt{3}}{2}t\right) \\ 2 \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$



$$e^{-\frac{\sqrt{3}}{2}t} \begin{bmatrix} 3 - \sqrt{3}i \\ 2 \end{bmatrix} \left( \cos\left(\frac{\sqrt{3}}{2}t\right) + i \sin\left(\frac{\sqrt{3}}{2}t\right) \right) = e^{-\frac{\sqrt{3}}{2}t} \begin{pmatrix} -3 \cos\left(\frac{\sqrt{3}}{2}t\right) - 3i \sin\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}i \cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}t\right) \\ 2 \cos\left(\frac{\sqrt{3}}{2}t\right) + i 2 \sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

7. Verify that  $\Psi = \begin{pmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{pmatrix}$  is a solution to  $\vec{x}' = \begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix} \vec{x}$ . Does the solution represent a fundamental set? (8 points)

$$\Psi' = \begin{pmatrix} -6e^{-2t} & e^t & 3e^{3t} \\ 4e^{-2t} & -e^t & -3e^{3t} \\ -4e^{-2t} & e^t & 0 \end{pmatrix}$$

$$\begin{pmatrix} -8 & -11 & -2 \\ 6 & 9 & 2 \\ -6 & -6 & 1 \end{pmatrix} \begin{pmatrix} 3e^{-2t} & e^t & e^{3t} \\ -2e^{-2t} & -e^t & -e^{3t} \\ 2e^{-2t} & e^t & 0 \end{pmatrix} =$$

$$\begin{pmatrix} -24e^{-2t} + 22e^{-2t} - 4e^{-2t} & -8e^t + 11e^t - 2e^t & -8e^{3t} + 11e^{3t} + 0 \\ 18e^{-2t} - 18e^{2t} + 4e^{2t} & 6e^t - 9e^t + 2e^t & 6e^{3t} - 9e^{3t} + 0 \\ -18e^{-2t} + 12e^{2t} + 2e^{-2t} & -6e^t + 6e^t + e^t & -6e^{3t} + 6e^{3t} + 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6e^{-2t} & e^t & 3e^{3t} \\ 4e^{-2t} & -e^t & -3e^{3t} \\ -4e^{-2t} & e^t & 0 \end{pmatrix}$$

8. Solve  $\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \vec{x}$  for the general solution (with real terms only). Describe the behavior of the origin: a repeller, attractor, or saddle point. (8 points)

$$(1-\lambda)(7-\lambda) + 9$$

$$\lambda^2 - 8\lambda + 7 + 9 =$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

$$\lambda = 4$$

$$\begin{bmatrix} 1-4 & -3 \\ 3 & 7-4 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}$$

$$\begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} -3 & -3 & -1 \\ 3 & 3 & 1 \end{array} \right] \Rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 1/3 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 = -x_2 + 1/3$$

$$x_2 = x_2 + 0$$

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} t e^{4t} + \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} e^{4t}$$

origin is a repeller