

Instructions: Show all work. Give exact answers unless specifically asked to round. Be sure to answer all parts of each question.

1. Write a system of equations that models a 2-tank system where Tank A has 1000L of brine solution containing 100kg of salt to start, and Tank B has 2000L with no salt to start. Pure water flows into Tank A at 4L/s and the mixture flows into Tank B at the same rate.

$$\frac{dA}{dt} = -\frac{A}{1000} \cdot 4 = -\frac{A}{250}$$

$$A(0) = 100 \\ B(0) = 0$$

$$\frac{dB}{dt} = \frac{A}{250} - \frac{B}{2000} \cdot 4 = \frac{A}{250} - \frac{B}{500}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}' = \begin{bmatrix} -\frac{1}{250} & 0 \\ \frac{1}{250} & -\frac{1}{500} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

2. Estimate the solution to $y' = 2xy$, $y(0) = 2$ at the point $y(1)$ in 3 steps using improved Eulers:

$$k_1 = f(t_n, y_n)$$

$$u_{n+1} = y_n + \Delta t k_1$$

$$k_2 = f(t_{n+1}, u_{n+1})$$

$$y_{n+1} = y_n + \frac{1}{2} \Delta t (k_1 + k_2)$$

$$\Delta x = \frac{1}{3}$$

Compare your result to the true solution $y = 2e^{x^2}$.

$$x_0 = 0 \quad y_0 = 2 \quad k_{10} = 2(0)(2) = 0 \quad u_1 = 2 + \frac{1}{3}(0) = 2$$

$$k_{20} = 2(0)(2) = 0 \quad y_1 = 2 + \frac{1}{2} \cdot \frac{1}{3} (0 + 0) = 2$$

$$x_1 = \frac{1}{3} \quad y_1 = 2 \quad k_{11} = 2\left(\frac{1}{3}\right)(2) = \frac{4}{3} \quad u_2 = 2 + \frac{1}{3}\left(\frac{4}{3}\right) = \frac{22}{9}$$

$$k_{21} = 2\left(\frac{1}{3}\right)\left(\frac{22}{9}\right) = \frac{44}{27} \quad y_2 = 2 + \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{4}{3} + \frac{44}{27}\right) = \frac{202}{81}$$

$$x_2 = \frac{2}{3} \quad y_2 = \frac{202}{81} \quad k_{12} = (2)\left(\frac{2}{3}\right)\left(\frac{202}{81}\right) = \frac{808}{243} \quad u_3 = \frac{202}{81} + \frac{1}{3}\left(\frac{808}{243}\right) = \frac{4626}{729}$$

$$k_{22} = 2\left(\frac{2}{3}\right)\left(\frac{4626}{729}\right) = \frac{10504}{2187} \quad y_3 = \frac{202}{81} + \frac{1}{2} \cdot \frac{1}{3} \left(\frac{808}{243} + \frac{10504}{2187}\right) =$$

$$x_3 = 1 \quad y_3 = \frac{25250}{6561} \approx 3.848$$

$$y(1) = 5.43656 \dots \approx 2e^1$$