CHARACTERISTIC EQUATIONS

Solving differential equations typically involves finding techniques that will work in particular cases rather than a general procedure for all types of situations. Some of these techniques have very limited applications. One of the broadest applications of solution techniques involves finding a characteristic equation, created from assuming a solution to a problem, usually of the form of an exponential. This technique applies readily to differential equations of all orders, assuming that the equation is of the right type. An equation with constant coefficients is most typical for the exponential case, but we will explore other situations where a similar procedure can work when the equation does not have constant coefficients.

Example 1. For the differential equation $y'' + 3y' - 4y = 0$, find the characteristic equation for the problem, and solve for y(t). Assume the solution is of the form $y(t) = Ae^{rt}$.

Assuming this solution form, take some derivatives and plug them into the differential equation.

$$y'(t) = Are^{rt}, y''(t) = Ar^2e^{rt}$$
$$y'' + 3y' - 4y = Ar^2e^{rt} + 3Are^{rt} - 4Ae^{rt} = 0$$

Since Ae^{rt} is in every term, factor it out.

$$Ae^{rt}(r^2 + 3r - 4) = 0$$

If A=0, we have only the trivial solution (i.e. y(t)=0 everywhere), and e^{rt} is never zero. That leaves us with $r^2 + 3r - 4 = 0$ as a means of obtaining the solution. If the equation is factorable, factor it. If not, you can use the quadratic formula to find real or complex roots. This equation is called the **characteristic equation**.

Here, we can factor $r^2 + 3r - 4 = (r + 4)(r - 1) = 0$, and so our solution of the form $y(t) = Ae^{rt}$, will work as long as r=-4, or r=1. Thus our general solution for $y(t) = Ae^{-4t} + Be^{t}$. The coefficients can be determined by the initial conditions. Sometimes this solution is notated as $y_c(t)$ since it's the solution to the characteristic equation.

Any time we have constant coefficients, we can use this method on homogeneous differential equations of any order, including first order, and higher order problems.

Example 2. For the differential equation $y' + ay = 0$, does the method used in Example 1 work?

It does work if a is a constant. Assume the solution $y(t) = Ae^{rt}$ and plug it and its derivative $y'(t) = Are^{rt}$ into the equation.

$$y' + ay = Are^{rt} + aAe^{rt} = 0$$

Factor out the common terms to get $Ae^{rt}(r + a) = 0$. The same conditions a before hold. A=0 leaves us only with y(t)=0, the trivial solution, and the exponential can never be zero, thus, r= -a is the only solution. So, $y(t) = Ae^{-at}$.

Example 3. For the differential equation $y''' + 2y'' - 9y' - 18y = 0$, find the characteristic equation for the problem, and solve for y(t). Assume the solution is of the form $y(t) = Ae^{rt}$.

For higher order problems like this, the great difficulty is in factoring the polynomial produced. However, even if it's not factorable, we have numerical methods (such as using our calculator or Newton's method) for finding the zeros of such a function, unless we are lucky, and it factors more easily.

Our derivatives are:

$$y'(t) = Are^{rt}, y'(t) = Are^{rt}, y''(t) = Ar^2e^{rt}, y'''(t) = Ar^3e^{rt}$$

These give us the characteristic equation:

$$y''' + 2y'' - 9y' - 18y = Ar^3e^{rt} + 2Ar^2e^{rt} - 9Are^{rt} - 18Are^{rt} = 0$$
$$Ae^{rt}(r^3 + 2r^2 - 9r - 18) = Ae^{rt}[r^2(r + 2) - 9(r + 2)] = Ae^{rt}(r^2 - 9)(r + 2) =$$
$$Ae^{rt}(r + 3)(r - 3)(r + 2) = 0$$

Our equation here factored by grouping, giving us three real solutions. So $y(t) = Ae^{-3t} + Be^{3t} + Ce^{-2t}$, which is just the sum of all the separate solutions.

All of our examples so far have had distinct real roots. Let's look at a situation with repeated roots.

Example 4. Solve the differential equation $y'' + 4y' + 4y = 0$ for the general solution. As we've seen happen in the past, we can transform this differential equation into a polynomial. Each derivative gets a factor of r, and the original function, being just a multiple of the assumed solution, is treated like a constant. It will save time to recognize this pattern and jump directly to the characteristic equation.

$$r^2 + 4r + 4 = (r + 2)^2 = 0$$

This equation as a repeated root at r= -2. For each order of a differential equation, we need that many distinct solutions, i.e. first order has one, second order two, third order three, etc. But it appears at first that since our root is repeated we have only one. The solution to this is to use the value of r as one solution: $y_1(t) = e^{-2t}$, and to create the other solution by multiplying this solution by t: $y_2(t) = te^{-2t}$. This will create a second fundamental solution. We can check this in the Wronskian.

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}$$

This exponential is never equal to zero, so it does form a fundamental set. Thus, our general solution is $y(t) = Ae^{-2t} + Bte^{-2t} = (A + Bt)e^{-2t}$.

We may also have to deal with situations involving complex roots of the form $r = \lambda \pm \mu i$. I'll do two examples below, one involving a purely imaginary root (λ=0), and one with λ≠0.

Example 5. Solve the differential equation $y'' + 4y = 0$ for the general solution. Here, our characteristic equation is $r^2 + 4 = 0$, which has solutions of $r^2 = -4 \rightarrow r = \pm 2i$.

Our equations are describing behaviour in the real world, so rather than using complex exponentials of the form $y(t) = Ae^{-2it} + Be^{2it}$, we will use Euler's formula to convert these into sines and cosines.

Given that $e^{ix} = \cos(x) + isin(x)$, we can solve for sine and cosine using complex exponentials as:

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$
$$\cos(t) = \frac{e^{it} + e^{-it}}{2}$$

Given this relationship, we can represent our real solutions as $y(t) = Acos(2t) + Bsin(2t)$. For any characteristic equation with a solution of the form $e^{\pm\mu it}$ we will end up with an equivalent sent of fundamental solutions as $\cos(\mu t) + \sin(\mu t)$. By choosing coefficients correctly (possibly complex coefficients), we can always make this conversion.

We can also check the Wronskian to ensure these these do form a fundamental set of solutions:

$$W = \begin{vmatrix} cos2t & sin2t \\ -2sin2t & 2cos2t \end{vmatrix} = 2cos^2 2t + 2sin^2 2t = 2 \neq 0$$

Example 6. Solve the differential equation $y'' + 2y' + 11y = 0$ for the general solution. In the more general case, we will have to use the quadratic formula to obtain solutions to the characteristic equation, here $r^2 + 2r + 11 = 0$, and often, especially in application problems, the results will not be pretty whole numbers. They will be ugly fractions with square roots, as they will be here.

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(11)}}{2(1)} = \frac{-2 \pm \sqrt{-40}}{2} = \frac{-2 \pm 2\sqrt{10}i}{2} = -1 \pm \sqrt{10}i$$

The arithmetic for the breakdown is as follows:

$$e^{(-1+\sqrt{10}i)t} + e^{-1-\sqrt{10}i} = e^{-1t}e^{\sqrt{10}it} + e^{-1t}e^{-\sqrt{10}it} = e^{-t}(e^{\sqrt{10}it} + e^{-\sqrt{10}it})$$

The bit in the parentheses is what is converted to sine and cosine using Euler's formula.

$$e^{-t}(e^{\sqrt{10}it} + e^{-\sqrt{10}it}) = e^{-t}(cos\sqrt{10}t + sin\sqrt{10}t)$$

Notice that the real part stays in the exponential, and the imaginary part (without the i) goes into the trig functions. The general form of the solution then, for a complex exponential is for $r = \lambda \pm \mu i$: $y_c(t) = e^{\lambda t}(Acos(\mu t) + Bsin(\mu t)) = Ae^{\lambda t}\cos(\mu t) + Be^{\lambda t}\sin(\mu t)$.

Practice Problems.

Find the characteristic equation for each differential equation and find the general solution. Some of the higher-order problems may be difficult to factor. If you can find one or more real root from your calculator (or from factoring), you can reduce the problem by long division to get any remaining complex roots from the quadratic formula. For all these problems, assume a solution of $y(t) = e^{rt}$. The Roman numeral indicate higher order derivatives above three, i.e. IV is the fourth derivative.

1. $y' + 2y = 0$
2. $2y' - 3y = 0$
3. $y'' - 7y' + 12y = 0$
4. $y'' + 3y' + 2y = 0$
5. $y'' - 2y' - 25y = 0$
6. $y'' + 2y - 7y = 0$
7. $y'' + y = 0$
8. $4y'' - 9y = 0$
9. $y'' - 8y' + 16y = 0$
10. $y'' + 2y' + 10y = 0$
11. $y'' - 5y' + 25y = 0$
12. $5y'' + 6y' + 8y = 0$
13. $25y'' + 70y' + 49y = 0$
14. $y''' + y' = 0$
15. $y''' + 2y'' - y' - 2y = 0$
16. $y^{IV} + y'' = 0$
17. $y^{IV} + 2y''' + y'' = 0$
18. $y^{IV} + y''' - 7y'' - y' + 6y = 0$

19. $y''' - y = 0$
20. $y^{VI} - y'' = 0$
21. $y^{VIII} + 8y^{IV} + 16y = 0$

Characteristic equations can also result from polynomial solutions, and polynomial coefficients of differential equations. This general method is called Cauchy-Euler equations. For instance, since we know that when we take derivatives, each polynomial loses powers by 1 for each derivative. Thus if we multiply each successive derivative by power of t, increasing with each order of derivative, then we can get an equation we can solve by assuming a solution of $y(t) = t^n$. For instance, we're talking about equations like:

$$aty' + by = 0$$
$$at^2y'' + bty' + cy = 0$$
$$at^3y''' + bt^2y'' + cty' + dy = 0$$

And so forth.

Example 7. Solve the differential equation $2t^2y'' + 3ty' - y = 0$ by assuming the solution $y(t) = t^n$.

As with the exponential solution, start by taking derivatives and plugging them into the equation: $y'(t) = nt^{n-1}, y''(t) = n(n-1)t^{n-2}$.

$$2t^2y'' + 3ty' - y = 2t^2n(n-1)t^{n-2} + 3tnt^{n-1} - t^n = 0$$

Combining the powers of t, this reduces to $2n(n-1)t^n + 3nt^n - t^n = 0$, and then factor out the common t^n: $t^n(2n(n-1) + 3n - 1) = 0$, which further simplifies to $2n^2 - 2n + 3n - 1 = 2n^2 - n - 1 = 0$. This is the characteristic polynomial of this equation, and we can use it to solve for the powers of n that will satisfy the equation.

$$2n^2 - n - 1 = (2n+1)(n-1) = 0$$

Thus, n=1, and n= -1/2. So our solution is $y(t) = At + Bt^{-\frac{1}{2}} = At + \frac{B}{\sqrt{t}}$.

Obtaining complex solutions from these equations are harder to deal with. Repeated roots should be dealt with by using reduction of order on one solution obtained in this way.

Practice Problems.

For each of the problems below, find the general solution to the differential equation. Assume a solution of $y(t) = t^n$.

1. $t^2y'' + 5ty' + 4y = 0$
2. $t^2y'' + 4ty' + 2y = 0$
3. $t^2y'' - 4ty' + 4y = 0$
4. $t^2y'' - 4ty' + 6y = 0$
5. $t^2y'' + 3ty' + y = 0$

Other assumed solutions (also called an Ansatz ("guess") solution) can work when predictable patterns are available. The two types provided here are the most common. Figuring out what those patterns imply about the form of the solution is an art best developed through experience.