SOLVING (FIRST ORDER) HOMOGENEOUS EQUATIONS

"Homogeneous" means different things in different types of differential equations. In higher order equations, it refers to the absence of a forcing function. In partial differential equation it refers to having zero on the boundary conditions. But in first order equations, it refers to a very specific type of equation wherein it's possible to solve the equation according to the ratio of the function and the independent variable. To solve these types of problems we will have to first recognize that the problem is of this type, and then make a correct substitution that will allow us to separate the variables.

1. Recognizing a first order homogeneous ordinary differential equation.

First order homogeneous equations tend to come in two forms. They are mathematically equivalent to each other, but look different because they are solved for different things.

Version A: Differential form

$$M(x,y)dx + N(x,y)dy = 0$$

Version B: dy/dx form

$$\frac{dy}{dx} = f(x,y), y' = -\frac{M(x,y)}{N(x,y)}$$

As you can see, Version A can be solved for the second form of Version B, so while the first example of version B is accurate, we tend to see the f(x,y) function as a ratio. The key will be if we can make a substitution that will make the problem separable. Many types of differential equations can be written in this form, including ones that are separable without a substitution, and ones that are exact (and can't be solved this way).

Supposing that the differential equation is not already separable, to show that the problem is homogeneous and not exact, we can test it with the following substitution: $x \to tx, y \to ty$. Replace x with tx and y with ty. If the function is homogeneous, we'll be able to factor out the same degree of t from every term in both M and N, and end up with $t^n M$ and $t^n N$ everywhere M and N used to be. They need to be same n for both equations, and there can be no t's left behind. This is a clever way of determining if we can write the problem in terms of y/x without actually writing it that way.

We can expect this to work if all the polynomial terms are the same degree, or if any non-polynomial terms are already written as y/x.

(Incidentally, it's possible we could have x/y as well, but this is less common.)

Example 1. Determine if the differential equation $(x^3 + y^3)dx + (x^2y + y^2x)dy = 0$ is homogeneous.

Replace $x \to tx, y \to ty$ and we get: $((tx)^3 + (ty)^3)dx + ((tx)^2ty + (ty)^2tx)dy = 0 \to$
$(t^3x^3 + t^3y^3)dx + (t^2x^2ty + t^2y^2tx)dy = 0 \to t^3(x^3 + y^3)dx + t^3(x^2y + y^2x)dy = 0$

Notice how we factored t^3 out of both terms? That our evidence that the problem is homogeneous, and specifically, it's homogeneous of degree three. (The degree of a homogeneous problem is the power of the t's factored out.)

This is equivalent to writing the equation in terms of y/x. Since our highest power is x^3, divide everything by that.

$$\frac{(x^3+y^3)}{x^3}dx + \frac{(x^2y+y^2x)}{x^3}dy = 0 \rightarrow \left(1+\left(\frac{y}{x}\right)^3\right)dx + \left(\frac{y}{x}+\left(\frac{y}{x}\right)^2\right)dy = 0$$

You can choose which method you prefer. For the degree here, you can use the highest power of x you divided by to get all the terms as functions of y/x.

Similarly, we can look at an example solved for y' and do something similar.

Example 2. Determine if the differential equation $y' = \frac{x-y}{x+ysin(\frac{y}{x})}$ is first order homogeneous.

Replace $x \rightarrow tx, y \rightarrow ty$: $y' = \frac{tx-ty}{tx+tysin(\frac{ty}{tx})} = \frac{t(x-y)}{t(x+ysin(\frac{y}{x}))}$

Notice how the t's cancel in this form? This is homogeneous of degree 1. We can also solve this for the y/x form by dividing everything by x:

$$y' = \frac{x-y}{x+ysin(\frac{y}{x})} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}\sin(\frac{y}{x})}$$

2. Solving a first order homogeneous equation

Once we've gotten the proof that the equation is homogeneous, we can solve the equation by making a substitution y=vx where v is an unknown function of x. (Or, if you solved the equation into the second form in Example 1 in terms of y/x, let v=y/x.) We will also need to find dy in terms of dx. Here, which version we use will depend on whether we are using Version A (Example 1) or Version B (Example 2).

Use the product rule or implicit differentiation to find dy or y'.

In differential form: y=vx becomes dy=xdv+vdx.
In prime form: y=vx becomes y'=xv' + v (since dx/dx=1).

Example 3. Solve the differential equation $(x^3+y^3)dx + (x^2y+y^2x)dy = 0$.

Solve the equation for dy/dx.

$$y' = \frac{x^3+y^3}{x^2y+y^2x}$$

Replace y=vx, and y'=xv'+v

$$xv' + v = \frac{x^3 + v^3x^3}{x^2vx + x^2v^2x}$$

Factor out the x^3 and put the extra v on the other side.

$$xv' + v = \frac{x^3(1 + v^3)}{x^3(v + v^2)} = \frac{(1 + v^3)}{v(1 + v)}$$
$$xv' = \frac{(1 + v^3)}{(v + v^2)} - v = \frac{1 + v^3 - v(v - v^2)}{v + v^2} = \frac{1 + v^3 - v^2 - v^3}{v + v^2} = \frac{-v^2 + 1}{v + v^2}$$

Now separate the variables by putting all the v's on the left and the x's on the right.

$$\frac{v + v^2}{-v^2 + 1}dv = \frac{1}{x}dx$$

Now integrate both sides. The right side is easy. But the left side is typically more complicated and may require inverse trig functions, trig substitution, partial fractions, long division of polynomials or other techniques. In this case, the problem factors and reduces, but we will still need long division.

$$\frac{v(1 + v)}{(1 - v)(1 + v)}dv = \frac{v}{1 - v} = \frac{-v}{v - 1} = \frac{1}{x}dx$$

$$\left(-1 - \frac{1}{v - 1}\right)dv = \frac{1}{x}dx$$

$$-v - \ln|v - 1| = \ln|x| + C$$

It's often impossible to solve for y in these problems, so we are left with an implicit equation. But we do have to replace v=y/x in order to get back to our original variables.

$$-\frac{y}{x} - \ln\left|\frac{y}{x} - 1\right| = \ln|x| + C$$

Example 4. Solve the homogeneous differential equation $y' = \frac{1-\frac{y}{x}}{1+\frac{y}{x}\sin(\frac{y}{x})}$. This equation has already been solved for the form y/x, so substitute y/x=v. And y'=xv'+v.

$$xv' + v = \frac{1 - v}{1 + v\sin(v)}$$

At this point, all the extra x's are already cancelled out on the right side since we did this part of the algebra before. Collect all the v's and separate the variables.

$$xv' = \frac{1 - v}{1 + v\sin(v)} - v = \frac{1 - v - v(1 + vsinv)}{1 + vsin(v)} = \frac{1 - 2v - v^2sinv}{1 + vsinv}$$

$$\frac{1 + v sinv}{1 - 2v - v^2 sinv} dv = \frac{1}{x} dx$$

Textbook problems are chosen so that these integrals are relatively easy to do, or at least *doable*. In the real world, we often find that things are more complicated. This is as far as we can get with this one.

$$\int_{\frac{y_0}{x_0}}^{\frac{y}{x}} \frac{1 + v sinv}{1 - 2v - v^2 sinv} dv = \ln|x| + C$$

Practice Problems. Prove that the differential equation is homogeneous. State the degree. Then solve the equation using an appropriate substitution.

1. $(x - y)dx + xdy = 0$
2. $(y^2 + yx)dx - x^2 dy = 0$
3. $\frac{dy}{dx} = \frac{x+3y}{3x+y}$
4. $-ydx + \left(x + \sqrt{xy}\right)dy = 0$
5. $x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$
6. $\left(x + ye^{\frac{y}{x}}\right)dx - xe^{\frac{y}{x}}dy = 0, y(1) = 0$
7. $ydx + x(lnx - lny - 1)dy = 0, y(1) = e$
8. $y' = \frac{x+y}{2x}$
9. $y' = \frac{xy}{x^2-y^2}$
10. $y' = \frac{2x+3y}{x}$
11. $\left(xsec\left(\frac{y}{x}\right) + y\right)dx - xdy = 0, y(1) = 0$