Concentration Problems

Concentration problems, which I usually refer to as 'tank problems' because they involve mixtures being pumped into and out of tanks, can we generally thought of in terms of a simple relation:

$$\frac{dA}{dt} = Rate_{in} - Rate_{out}$$

We do need to be careful to check the units on our final equation to make sure all the terms end up being amount/time. Our rates aren't always given in these terms.

Example 1. A tank initially contains 400L of pure water. A mixture containing a concentration of 16g/L of sugar at a rate of 4 L/sec, and the well-stirred mixture leaves the tank at the same rate. Find an expression for the amount of sugar in the tank at any time t. What amount of sugar would be needed in the tank to maintain the same level of sugar in the tank over time?

The $Rate_{in}$ here is broken into two components: the amount of sugar per liter, and then number of liters per time. If we multiply the two, the liters will cancel and we'll get amount per time. (Our amount here is given in grams.)

$$Rate_{in} = \frac{16g}{L} \cdot \frac{4L}{sec} = \frac{64g}{sec}$$

$Rate_{out}$ is calculated similarly, but the amount of sugar going out of the tank is dependent on the amount A in the tank.

$$Rate_{out} = \frac{A\ (in\ grams)}{400L} \cdot \frac{4L}{sec} = \left(\frac{A}{100}\right)\frac{g}{sec}$$

Thus our equation becomes:

$$\frac{dA}{dt} = 64 - \frac{A}{100}$$

This is a first order differential equation that can be solved by separating the variables. It will make it a bit easier if we factor out the coefficient of the A first.

$$\frac{dA}{dt} = -\frac{1}{100}(A - 6400)$$

$$\frac{dA}{A - 6400} = -\frac{1}{100}dt$$

Integrate both sides.

$$\int \frac{dA}{A - 6400} = \int -\frac{1}{100}dt \rightarrow \ln|A - 6400| = -\frac{1}{100}t + C$$

Exponentiate both sides:

$$\ln|A - 6400| = -\frac{1}{100}t + C \rightarrow A - 6400 = e^{-\frac{1}{100}t+C} = Ke^{-\frac{1}{100}t}$$

Thus solving for A: $A(t) = 6400 + Ke^{-\frac{1}{100}t}$.

We'll use any initial conditions to solve for the constant K, which should be negative. Our initial condition is A(0)=0 since the tank is initially full of pure water.

$$0 = 6400 + Ke^0 \rightarrow K = -6400$$

So our final equation for A is: $A(t) = 6400 - 6400e^{-\frac{1}{100}t}$

We can say several things about this problem. For instance, since the exponential term goes to zero, we can tell the equilibrium value is 6400 g of sugar in the tank. This is the answer to the last question asked.

Example 2. A tank initially contains 400L of pure water. A mixture containing a concentration of 16g/L of sugar at a rate of 4 L/sec, and the well-stirred mixture leaves the tank at the rate of 3 L/sec. Find an expression for the amount of sugar in the tank at any time t. How long can this formula apply if the tank is half-full when we start?

This problem is similar to the first one, but now, the rate water is leaving the tank is slower than the rate water is coming into the tank. The difference tells us how fast that is happening: 4-3= 1 L/sec is being added to the tank.

Setting up our equation again:

$$Rate_{in} = \frac{16\ g}{L} \cdot \frac{4L}{sec} = 64\ g/sec$$

$$Rate_{out} = \frac{A\ (grams)}{(400 + 1t)L} \cdot \frac{3L}{sec} = \frac{3A}{400 + t}\ g/sec$$

The +t in the denominator is from the fact that the tank is filling by one liter per second since the rates of water in and water out are different.

Our equation then becomes:

$$\frac{dA}{dt} = 64 - \frac{3A}{400 + t}$$

This equation is linear of the form $A' + p(t)A = g(t)$, where $p(t) = \frac{3}{400+t}$, and $g(t) = 64$.

$$\frac{dA}{dt} + \left(\frac{3}{400+t}\right)A = 64$$

Solve this as you would any other linear equation. I'll use an integrating factor.

$$\mu = e^{\int \frac{3}{400+t}dt} = e^{3\ln|400+t|} = (400+t)^3$$

$$(400+t)^3\frac{dA}{dt} + 2(400+t)^2 A = 64(400+t)^3$$

$$((400+t)^3 A)' = 64(400+t)^3$$

$$(400+t)^3 A = \int 64(400+t)^3\,dt = \frac{64}{4}(400+t)^4 + C$$

$$A(t) = 16(400+t) + \frac{C}{(400+t)^3}$$

Since the initial value is still A(0)=0, we get:

$$0 = 6400 + \frac{C}{400^3} \to C = -4.096 \times 10^{11}$$

$$A(t) = 16(400+t) - \frac{4.096 \times 10^{11}}{(400+t)^3}$$

The amount of water in this tank is increasing, so it can't last forever. It can only last as long as the tank has room to hold the water. If the tank was half-full at the beginning, then the whole tank is 800 L. Since we are adding 1 L per sec to the tank, and there are 400 L left of the tank to fill, the equation will last 400 sec.

Practice Problems.

1. A pool contains 40,000 gallons of pure water. A pool cleaner wishes to add chlorine to the water at the rate of 5 ppm/gallon in water added at 5 gallons/minute. He achieves this by pumping water out of the pool at the same rate. Assuming that the chlorine is well-mixed in the pool, write a differential equation that represents the amount of chlorine in the pool in ppm (parts-per-million) at any given time t. Solve the equation and then determine how long it will take to get the entire pool to 15 ppm.
2. A 1500 gallon tank initially contains 1200 gallons of water with 5 lbs of salt dissolved in it. Water enters the tank at a rate of 8 gal/hr and the water entering the tank has a salt concentration of sin(t) lbs/gal. If a well-mixed solution leaves the tank at a rate of 6 gal/hr, how much salt is in the tank when it overflows?
3. A mixing tank initially contains 140 gallons of brine which contains 25 pounds of salt in solution. A new brine containing 1.5 pounds of salt per gallon begins entering the tank at the rate of 2 gal/minute while the well-stirred mixture leaves the tank at 1 gal/min. Assuming the mixture is kept uniform, find the amount of salt in the tank at the end of an hour.
4. A tank has pure water flowing into it at 20 L/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at the same rate. Initially, the tank contains 10 kg of salt in 100 L of water. Find the amount of salt in the tank at any time t.