Undetermined Coefficients

The method of undetermined coefficients is used to solve differential equations involving forcing terms, such as the f(t) in $ay'' + by' + cy = f(t)$. The solution to such a differential equation will involve the solution to the homogeneous equation $ay'' + by' + cy = 0$, which we call $y_c(t)$, and the solution to the forcing term, which we call $Y(t)$. This solution is sometimes called the *Ansatz* (the German word for "guess") because we form a "best guess" for the solution, and then solve for the coefficient that makes it work in the given problem, thus the term "undetermined coefficients".

The method of undetermined coefficients only works when the set of derivatives of the function is finite. For instance, if the derivatives will eventually repeat itself, or if they eventually become zero. Examples of functions of this type are sine and cosine (set of 2), the exponential (set of 1), hyperbolic sine and cosine (set of 2), and polynomials (set of n, where n is the highest degree of the forcing term). Undetermined coefficients will not work if the set of derivatives of the forcing term is infinite as in the case of rational or radical functions, the other 4 trig functions, etc. To deal with forcing terms of this type, you need to use the method variation of parameters.

If the forcing term does not look like the form of the solution to the homogeneous differential equation, one assumes a solution of the form of the f(t) function with an undetermined coefficient, together with any of its derivatives.

Forcing term doesn't match homogeneous solution. Assume Y(t) of the form:

Forcing term f(t)	**Ansatz Y(t)**
ke^{at}	Ae^{at}
$\mathrm{k}_1\sin(bt), k_2\cos(bt)$	$Acos(bt) + Bsin(bt)$
$k_1\sinh(at), k_2\cosh(at)$	$Acosh(at) + Bsinh(at)$
$k_1e^{at}\cos(bt), k_2e^{at}\sin(bt)$	$e^{at}(\mathrm{A}\cos(bt) + Bsin(bt))$
$a_nt^n + a_{n-1}t^{n-1} + \cdots + a_0$	$At^n + Bt^{n-1} + \cdots + Z$

Use all terms of the Ansatz even if only one term of the forcing function is present. For instance, for sin(t), you'll need to use both Acos(t) and Bsin(t). In the case of the polynomial, if the forcing term is t^3, you'll need to use all the terms down to the constant, $At^3 + Bt^2 + Ct + D$. The lower terms are needed to make sure the derivatives will cancel properly.

Example 1. What is the Ansatz for the differential equation $y'' + 4y' - 5y = 3\sin(t)$? Solve the homogenous for the fundamental solutions, first. The characteristic equation is $r^2 + 4r - 5 = 0 \rightarrow (r + 5)(r - 1) = 0 \rightarrow r = 1, r = -5$. So our $y_c(t) = Ce^t + De^{-5t}$. Neither of these are the form of sin(t). So our Ansatz is just $Y(t) = Acos(t) + Bsin(t)$. We'll need to solve this for the A and B constants first, before we solve for C and D in the homogeneous solution. To do this, we'll by taking some derivatives and plugging them into the original differential equation.

$$Y'(t) = -Asin(t) + Bcos(t), Y''(t) = -Acos(t) - Bsin(t)$$
$$y'' + 4y' - 5y = -Acos(t) - Bsin(t) - 4Asin(t) + 4Bcos(t) - 5Acos(t) - 5Bsin(t)$$
$$= 3\sin(t)$$

Now we collect the sine terms together in one equation, and the cosine terms in another.

$$-Bsin(t) - 4Asin(t) - 5Bsin(t) = 3\sin(t)$$
$$-Acos(t) + 4Bcos(t) - 5Acos(t) = 0\cos(t)$$

For these equations to be true, the coefficients on both sides must be equal. That leaves us with:

$$-B - 4A - 5B = 3 \rightarrow -4A - 6B = 3$$
$$-A + 4B - 5A = 0 \rightarrow -6A + 4B = 0$$

Solving the system yields $A = -\frac{3}{13}, B = -\frac{9}{26}$.

We can then use $y(t) = Ce^t + De^{-5t} = \frac{3}{13}\cos(t) - \frac{9}{26}\sin(t)$ to solve for any initial conditions provided and get C and D.

Things become a little more complicated when the forcing term is part of the solution to the homogeneous system. When that happens, you'll have to multiply the usual Ansatz by a power of t until you get a unique function that doesn't match one of the fundamental solutions.

Suppose we had the equation $y'' + 4y' - 5y = f(t)$, as in Example 1, but instead of f(t)=3sin(t), we instead had $f(t) = e^t$. Now we can't just use an Ansatz of Ae^t because that's already part of the homogeneous system, and so we know putting it through the derivative is only going to get us zero, by definition. What we need is a function that has a derivative of the form e^t, and that function is Ate^t, in fact, it's just what we do for repeated roots.

Example 2. What is the Ansatz for the differential equation $y'' + 4y = 2\sin(2t)$?
The characteristic equation for this problem is $r^2 + 4 = 0 \rightarrow r = \pm 2i$. This gives us $y_c(t) = Ccos(2t) + Dsin(2t)$. This is the form of the forcing function, so we can't use them again for Y(t). Instead we multiply by t to get a unique solution. $Y(t) = Atcos(2t) + Btsin(t)$. Taking derivatives will involve a product rule.

$$Y'(t) = Acos(2t) - 2Atsin(2t) + Bsin(2t) + 2Btcos(2t)$$
$$Y''(t) = -2Asin(2t) - 2Asin(2t) - 4Atcos(2t) + 2Bcos(2t) + 2Bcos(2t) - 4Btsin(2t)$$

Putting these into our differential equation we get:

$$y'' + 4y = -2Asin(2t) - 2Asin(2t) - 4Atcos(2t) + 2Bcos(2t) + 2Bcos(2t) - 4Btsin(2t) + 4Atcos(2t) + 4Btsin(2t) = 2\sin(2t)$$

The terms with t are going to cancel out, leaving us with just

$$-2Asin(2t) - 2Asin(2t) + 2Bcos(2t) + 2Bcos(2t) = 2\sin(2t)$$

And from here, we collect sine and cosine terms separately as before to solve for A and B.

$$-2Asin(2t) - 2Asin(2t) = 2\sin(2t)$$
$$2Bcos(2t) + 2Bcos(2t) = 0$$

$$-4A = 2$$
$$4B = 0$$

Y(t) turns out to be just $-\frac{1}{2}tsin(2t)$. We can use this information then to solve for the coefficients in the homogeneous equation if we are provided any initial conditions.

Practice problems.

Find the Ansatz of each equation. You will have to solve the characteristic equation first. Solve for the coefficients in the Ansatz. Solve for any initial conditions if they are provided.

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' - 2y' - 3y = 3\sin(t)$
3. $y'' - 2y' - 3y = 3e^{2t}\cos(t)$
4. $y'' - 2y' - 3y = 3te^{-t}$
5. $y'' + 9y = t^2 + 3\sin(t)$
6. $y'' + 9y = 5\sin(3t)$
7. $y'' + 2y = 4\sin(2t)$
8. $y'' + y = 3\sin(2t) + tcos(2t)$
9. $y'' + y' + 4y = 2\sinh(t)$
10. $y'' - y' - 2y = \cosh(2t)$
11. $y'' + y' - 2y = 2t, y(0) = 0, y'(0) = 1$
12. $y'' + 4y = 3\sin(2t), y(0) = 2, y'(0) = -1$
13. $y'' + y = t + tsin(t)$