

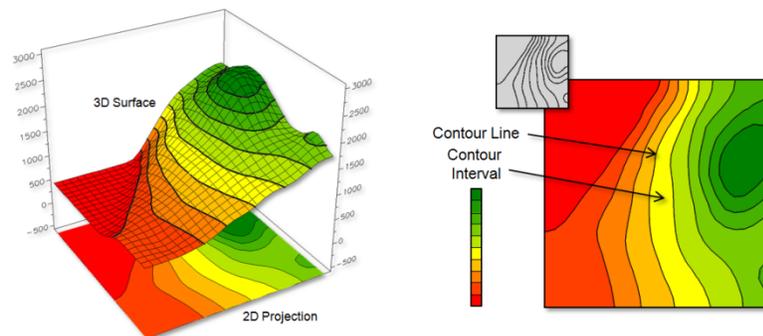
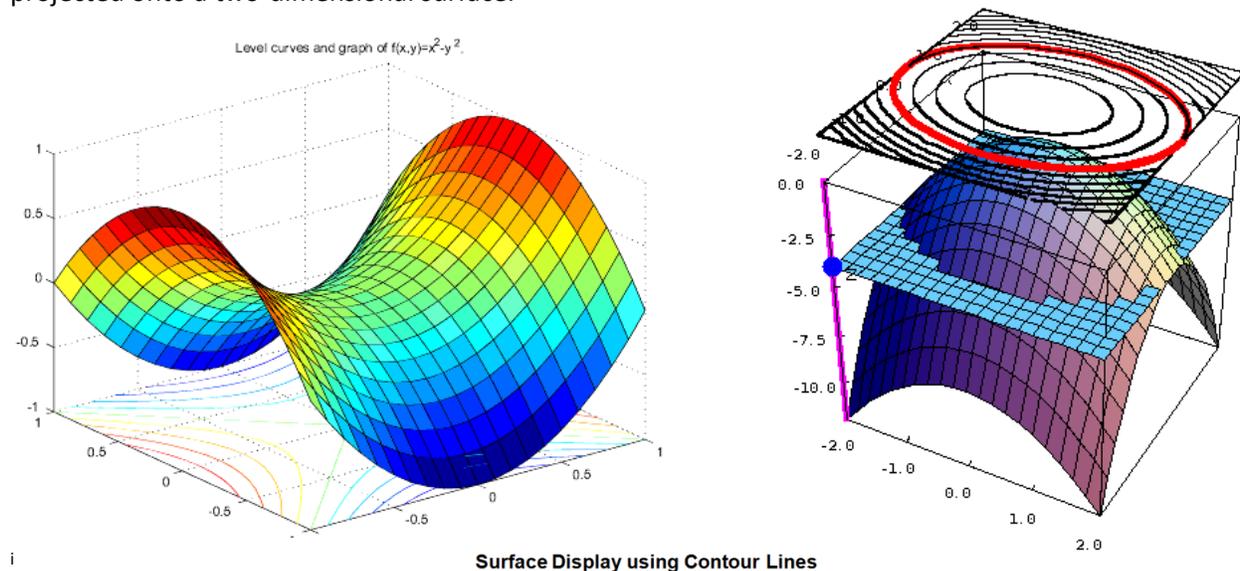
GRADIENTS AND LEVEL CURVES

There is a close relationship between level curves (also called contour curves or isolines) and the gradient vectors of a curve. Indeed, the two are everywhere perpendicular. This handout is going to explore the relationship between isolines and gradients to help us understand the shape of functions in three dimensions. This is a common application in physics when considering lines of force and the lines of equal force, or other similar applications in electricity and magnetism and other fields.

Let us first recall briefly how level curves (isolines) work as we translate from three dimensions into a two-dimensional graph.

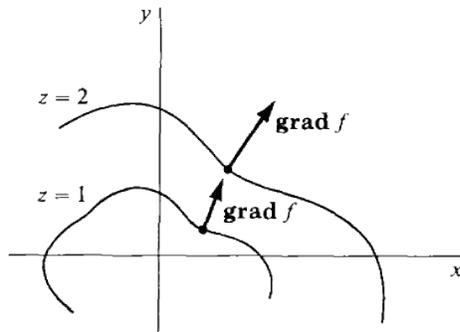
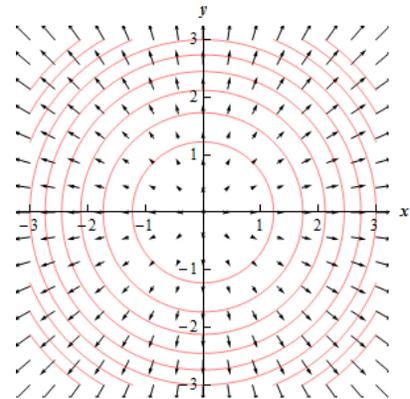
We start with a three-dimensional function. And we slice that function with a plane along specific values of one of the variables (typically the z-direction), and then project that intersection onto the two-dimensional plane. We repeat this process over and over at different levels of z to obtain a series of embedded curves that trace out the shape of the graph at fixed heights. If we know which direction z is increasing or decreasing in, we can in principle, uncollapse the graph to reconstruct the original function.

Below are displayed some examples of three dimensional curves and their associated level curves projected onto a two-dimensional surface.



Recall that the gradient of a function given by $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ represents the direction the surface is increasing it at a given point on the surface. Because gradients represent not only the rate of the increase, but also the direction of greatest increase, the gradient is always at right angles to the isolines where no change is taking place.

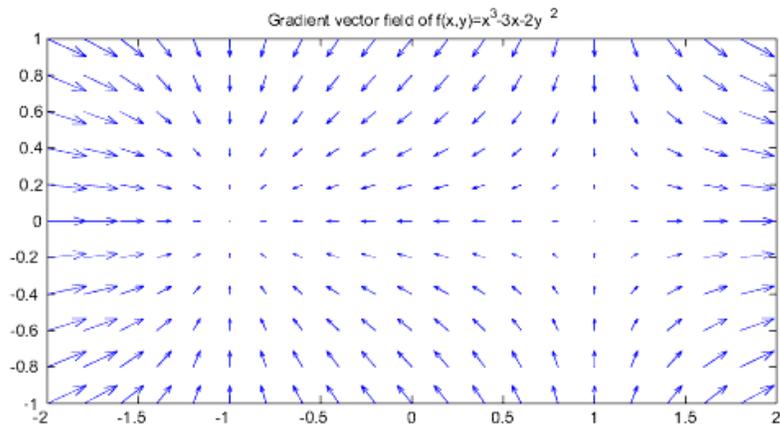
So consider for a paraboloid graph, whose level curves are circles, the gradient points radially outward from the origin. The relationship between the two is shown in the next graph. As the gradient, whose form for a paraboloid with a circular cross-section is $\langle 2ax, 2ay, 1 \rangle$, get closer to the origin they get shorter, and further away from the origin they get longer, but they always point in the same direction along any radial line from the origin, and so they are always perpendicular to the contour curves marking out the surface as it increases away from the center.



increase, but by themselves, they cannot tell us whether the curve is increasing or decreasing. It's the gradient that gives us that.

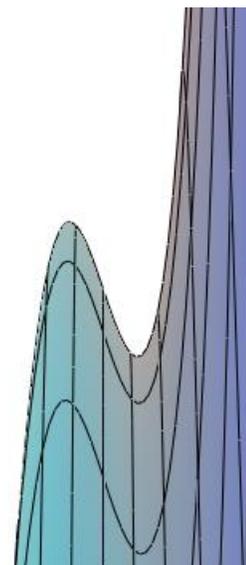
So consider the graph below showing the gradient only. What can you tell about the function it was derived from? Even without graphing the function we can see that there are two critical values on the graph. The vectors point in the direction of a point at $(-1,0)$. This is a maximum value since the gradient is increasing toward the point at the center. The second point at $(1,0)$ is a saddle point. Notice that the gradient is vertically moving toward the point (increasing), but horizontally is moving away from the point (decreasing).

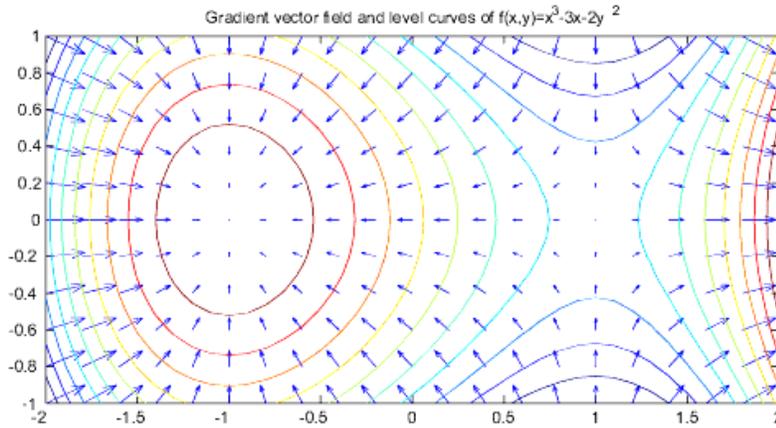
The gradient tells us the missing information about the third dimension, which is the direction of increase. The spacing of level curves themselves can give us information about the rate of increase. The more closely spaced they are, the faster the



Shown is the graph of the function with x as the horizontal axis, z as the vertical axis, and y is coming out of the paper.

It's also possible to work backwards from the gradients and obtain the level curves even without the original function. Since the gradient and isolines are mutually perpendicular, you can draw lines the connect gradients. The level curve lines for the gradient and function shown here are illustrated below.





Our goal for this handout will be to start with the gradient and sketch the general shape of the level curves (and vice versa). This will give us our most complete picture of a three dimensional graph without having to be an artist.

One way of proceeding is to begin with a function that we'd like to sketch, but this technique is most useful when we don't have the

function, and only the gradient or lines of force.

Example 1.

Sketch the gradient, and use that to draw the level curves, of the function $f(x,y) = x^3 + 3x^2 - xy + 2y^3 - 24y + 15$.

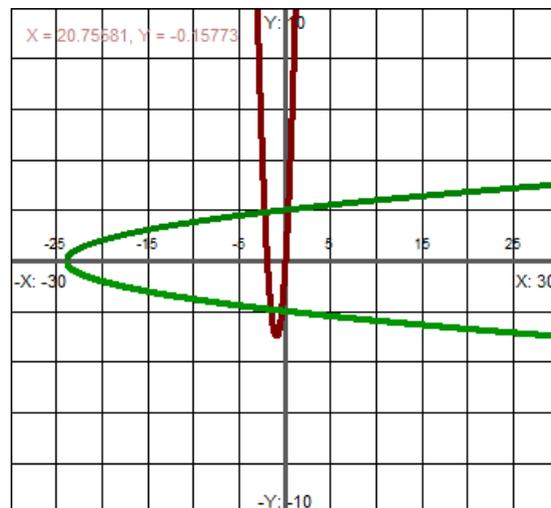
We begin by finding the gradient. $\nabla f = \langle 3x^2 + 6x - y, -x + 6y^2 - 24 \rangle$. Since the function is explicit, we need only concern ourselves here with the derivatives for x and y.

At this point, we could choose a series of points that mark out a grid for our plane and begin to draw gradient vectors. This is tedious to do by hand. We are most interested here in the primary features of the graph and it's general shape, so this level of detail is not required unless you have a computer program that can produce the graph for you. Instead, we are going to sketch the curves where each partial derivative is zero. This will divide the graph into several regions where the behaviour of the graph will be similar inside the region, and change as you cross the lines marking the critical lines.

For $\frac{\partial f}{\partial x} = 3x^2 + 6x - y$, when we set this equal to zero, solving for y is quite simple. $y = 3x^2 + 6x$.

When doing this by hand for $\frac{\partial f}{\partial y}$, you can solve for y and graph the function that way, or if using a graphing program as I am, solve for y. This may be more difficult, involve multiple square roots, involve completing the square, or other potential hazards. In

this case if $\frac{\partial f}{\partial y} = 0$, then $-x + 6y^2 - 24 = 0$, and thus $y = \pm \sqrt{\frac{x+24}{6}}$. And so we obtain the graph shown to the right.



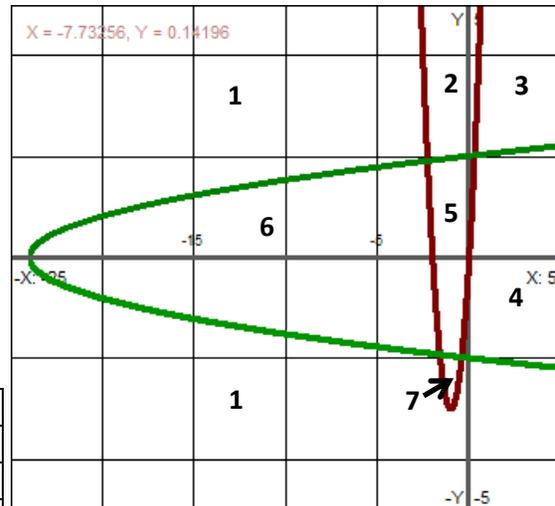
One thing we can immediately see from the graph is that there are four points of intersection. Since these lines represent the lines where the two partial derivatives are zero, we know that there are 4 possible extrema.

We can also see that the graph is divided into seven regions. The regions are numbered on the next graph.

For each one of these regions, we should choose a point to determine the direction of the gradient. This will give us a good overall picture of the function, and allow us to determine the character of the critical points.

The table shows the region, the points selected and the values of the gradients determined from those points.

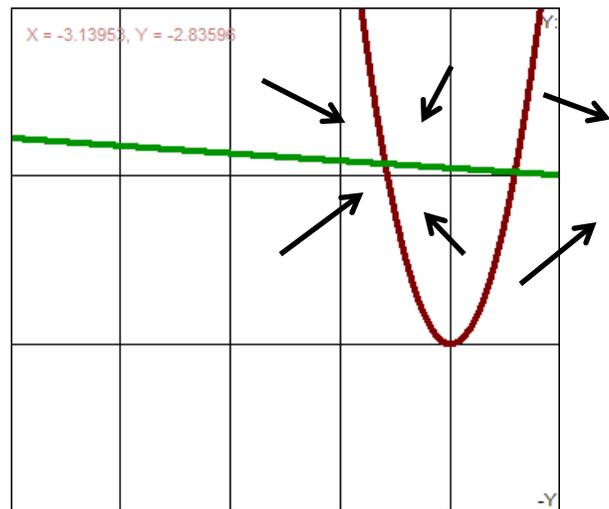
Region	x	y	∇f
1	-10	2	$\langle 238, 10 \rangle$
2	-1	4	$\langle -7, 73 \rangle$
3	5	4	$\langle 101, 67 \rangle$
4	5	0	$\langle 105, -29 \rangle$
5	-1	0	$\langle -3, -23 \rangle$
6	-10	0	$\langle 240, -14 \rangle$
7	-1	-2	$\langle -1, 1 \rangle$



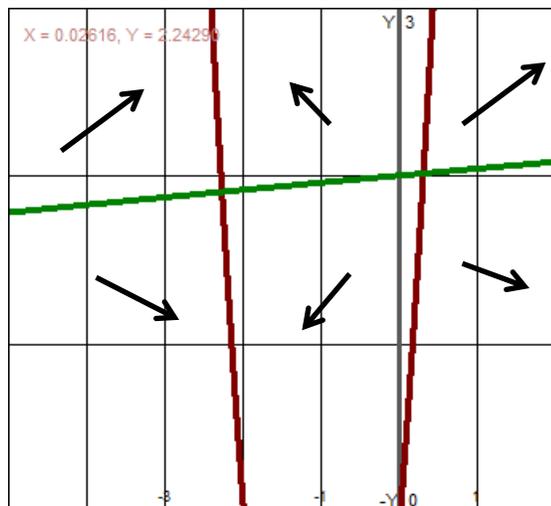
The magnitude of the vectors is not especially important to us here, but we are interested in the directions they are pointing in. Consider the graph below focused on region #7 in the graph above.

Notice that the gradient arrows are all pointing toward the leftmost intersection on this graph. This is going to be a maximum since the direction of the arrows is the directions of greatest increase.

The intersection of the right, however, is going to be a saddle point. The arrows are pointing the direction of the green line ($\frac{\partial f}{\partial y}$), but away from the red line ($\frac{\partial f}{\partial x}$). This means that the derivatives are of mixed sign here, and so we will have a saddle point.



Let's similarly zoom in on the other two intersections.



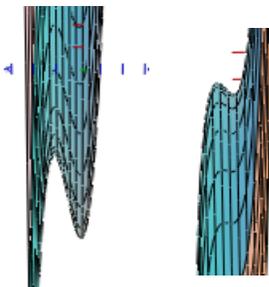
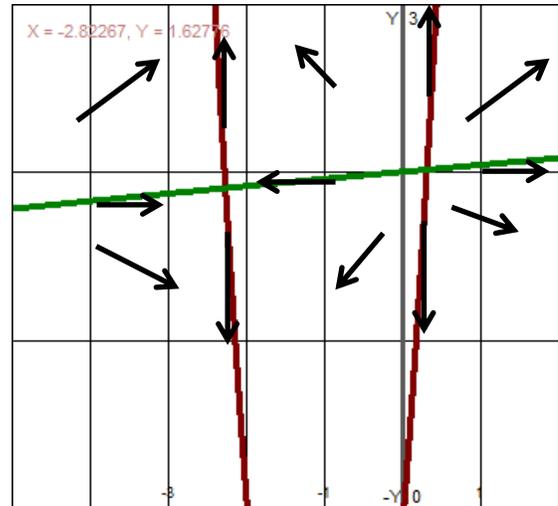
For the left intersection on this graph, the arrows are approaching the red line ($\frac{\partial f}{\partial x}$), but moving away from the green line ($\frac{\partial f}{\partial y}$). This is a saddle point. But the right intersection has all the arrows pointing away from it. This means that the intersection is a local minimum, since all directions of increase must point away from it.

To complete the picture of the gradient, recall that when $\frac{\partial f}{\partial y}$ is zero, the y-coordinate of the gradient vector is zero, and so the gradient must be vertical all along that line. Similarly, when $\frac{\partial f}{\partial x}$ is zero, the gradient has a zero x-coordinate, and so the gradient vectors must be horizontal.

We can use this information to sketch the level curves since, recall that they must be perpendicular to the gradients.

Around the right intersection, they are going to look like circles around the minimum. On the left side of the graph they are going to curve away from the saddle point.

If you can't visualize this, look at the graph at the top of page #3. The relationship of the extrema and the saddle point are flipped, but it will give you a good idea of what the level curves must be doing here.

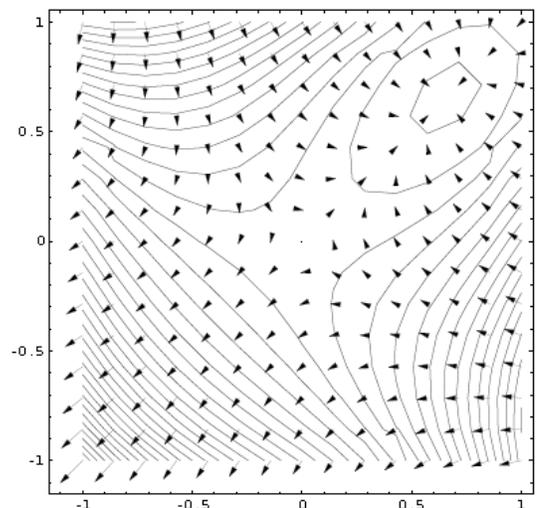


The graph on the left is the graph of the region around the minimum, and its nearby saddle point. The one on the right is the region around the maximum and its nearby saddle point.

One other thing to note about these types of graphs is that it's quite common for the gradient field to flip sign as it crosses the zero lines.

Notice that on the graph above that as you cross the red line ($\frac{\partial f}{\partial x}$), the vectors across the top three regions flips from $\langle +, + \rangle$ to $\langle -, + \rangle$ vectors, and then crossing the red line again, flips from $\langle -, + \rangle$ vectors to $\langle +, + \rangle$ again. Similarly, as the vectors cross the green line ($\frac{\partial f}{\partial y}$), the vectors flip from $\langle +, + \rangle$ along the left two regions, to $\langle +, - \rangle$ vectors. Unless the partial derivatives are perfect squares, this is typical behaviour. We can use this tendency when we don't have a function to check, but if we have the function, or at least the gradient vector, we should check explicitly rather than assume a particular behaviour.

Our goal for the next section is to achieve a graph like the one shown. For that, we will need a more detailed graph of the gradient field.

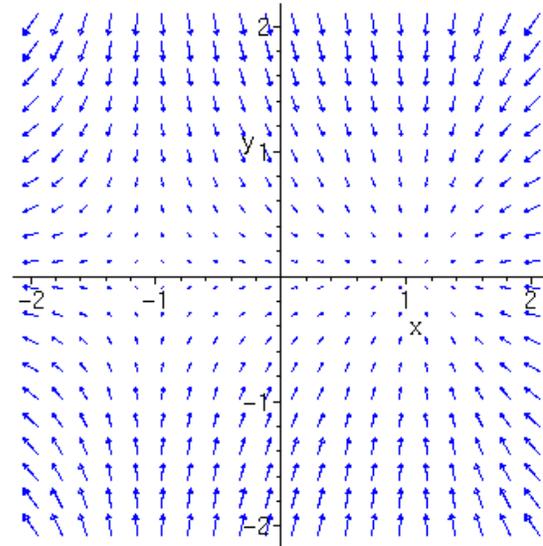


Example 2.

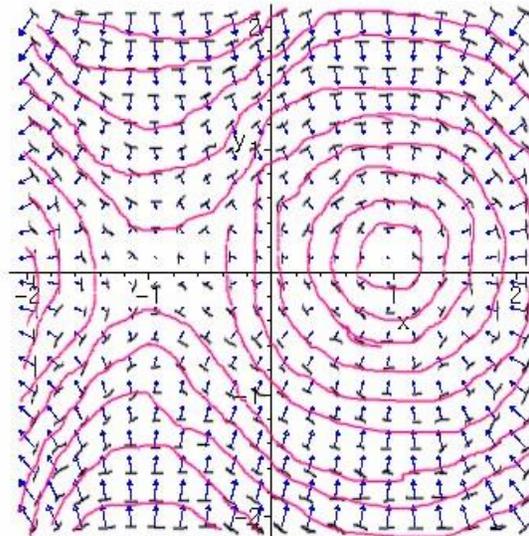
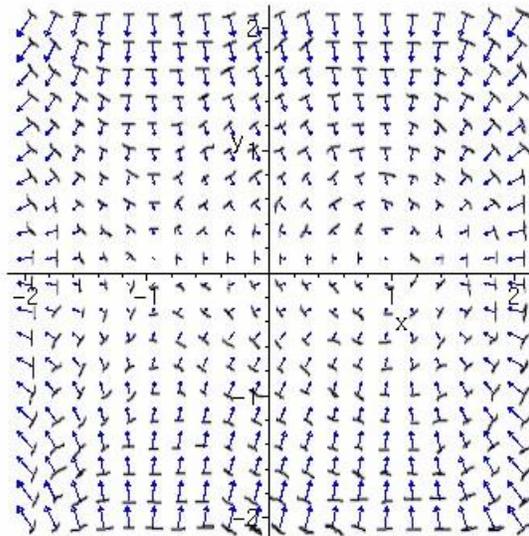
From a gradient field, sketch the level curves of the graph.

Consider the gradient field shown below. The key to drawing the isolines from the gradient field is to sketch perpendicular line segments at the end of each gradient vector, and then to connect those perpendicular segments into a complete curve.

The second graph below shows the same graph with the perpendicular segments at the end of each vector. And then the third graph shows the isolines sketched into complete curves. Unlike the example graph, I have not added additional isolines to represent the rate of increase, but you could place those into the graph easily by sketch additional curves between the existing ones.

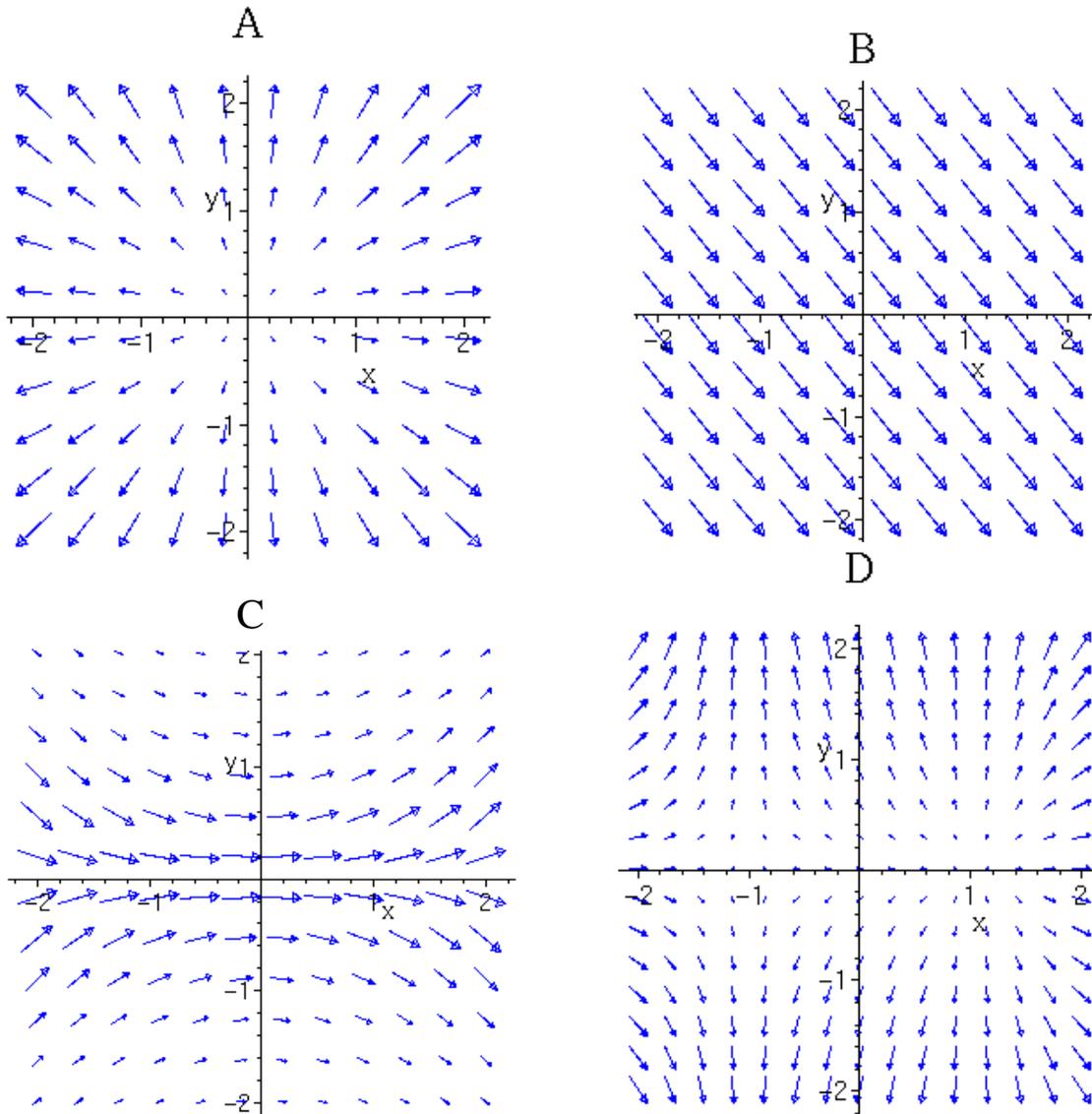


If you look carefully at the direction of the gradient field, the arrows are pointing toward the center of the circular feature on the right. This is a maximum. The feature on the left is a saddle point.

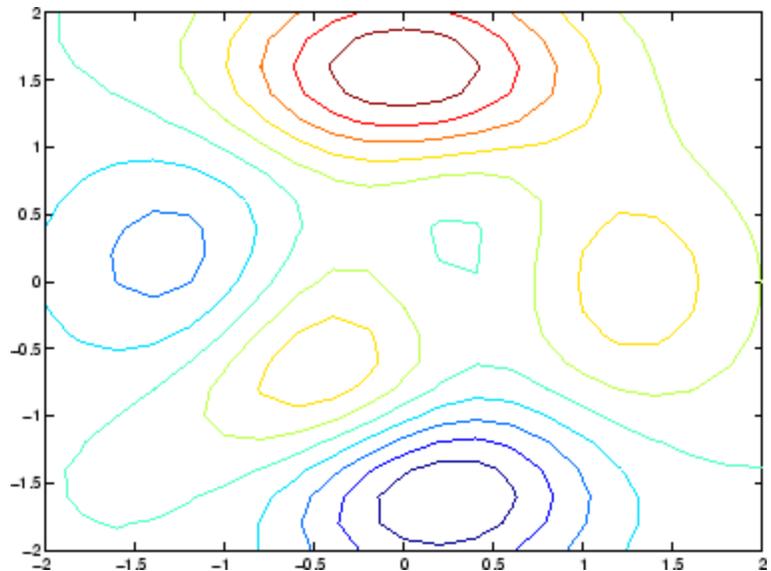


Practice Problems.

1. For each of the gradient graphs below, sketch the isolines. For each graph, if there are any critical points, determine whether each is a maximum, minimum or saddle point.



- Use the idea that gradients are perpendicular to level curves to sketch the gradient field for the graph below. Assume that blues are lower values, and reds are larger values to determine the direction of your vectors. Note any critical points or other interesting features, including saddle points.



3. Following the method in Example 1, for each of the following equations, analyze the graph by graphing the partial derivatives and analyzing the gradient field in each region of the plane.
- $f(x, y) = xy$
 - $g(x, y) = \sin(x) \cos(y)$
 - $h(x, y) = xe^y$
 - $k(x, y) = 4 - x^2 - 4y^2$
 - $m(x, y) = \frac{x}{1+x^2+y^2}$
 - $n(x, y) = x^2 e^{-xy/2}$
 - $p(x, y) = \frac{x+y}{x^2+y}$
 - $q(x, y) = 3x^3 - 12xy + y^3$
 - $r(x, y) = \frac{1}{x} + \frac{1}{y} + xy$
 - $s(x, y) = x^2 + 4xy + y^2 - 4x + 16y + 3$
 - $t(x, y) = 4x^2 - y$

ⁱ Figure 3 is by Joseph K. Berry <http://www.innovativegis.com/basis/mapanalysis/Topic18/Topic18.htm>