

202 Homework #7 key

(1)

1. to show orthogonal, find dot products. Orthogonal \Rightarrow independent.

$$\vec{u}_1 \cdot \vec{u}_2 = 6 - 6 + 0 = 0 \quad \vec{u}_2 \cdot \vec{u}_3 = 2 + 2 - 4 = 0 \quad \vec{u}_1 \cdot \vec{u}_3 = 3 - 3 + 0 = 0 \quad \checkmark$$

these do form a basis for \mathbb{R}^3 .

$$\|\vec{u}_1\| = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

$$\hat{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\hat{u}_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

$$\hat{u}_3 = \begin{bmatrix} 1/3\sqrt{2} \\ 1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$$

$$\|\vec{u}_2\| = \sqrt{4+4+1} = 3$$

$$\|\vec{u}_3\| = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$$

$\vec{u}_i \cdot \vec{u}_i = 1$ in orthonormal basis

$$\vec{x} \cdot \hat{u}_1 = 5 \cdot 1/\sqrt{2} - 3 \cdot 1/\sqrt{2} + 0 = 2/\sqrt{2}$$

$$\vec{x} \cdot \hat{u}_2 = 5 \cdot 2/3 - 3(2/3) + 1(-1/3) = 10/3 - 6/3 - 1/3 = 3/3 = 1$$

$$\vec{x} \cdot \hat{u}_3 = 5(1/3\sqrt{2}) - 3(1/3\sqrt{2}) + 1(4/3\sqrt{2}) = \frac{5}{3\sqrt{2}} - \frac{3}{3\sqrt{2}} + \frac{4}{3\sqrt{2}} = \frac{6}{3\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2/\sqrt{2} \\ 1 \\ 2/\sqrt{2} \end{bmatrix}$$

$$2. \vec{u}_1 \cdot \vec{u}_2 = -2 + 2 - 1 + 1 = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 1 + 2 - 2 - 1 = 0$$

$$\vec{u}_1 \cdot \vec{u}_4 = -1 + 2 + 1 - 2 = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = -2 + 1 + 2 - 1 = 0$$

$$\vec{u}_2 \cdot \vec{u}_4 = 2 + 1 - 1 - 2 = 0$$

$$\vec{u}_3 \cdot \vec{u}_4 = -1 + 1 - 2 + 2 = 0$$

$$\frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{4 + 10 - 3 + 3}{1 + 4 + 1 + 1} \Rightarrow \frac{14}{7} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{y}_{\parallel} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -6/7 \\ 3/7 \\ -3/7 \\ 3/7 \end{bmatrix} + \begin{bmatrix} 12/7 \\ 12/7 \\ -24/7 \\ 12/7 \end{bmatrix} = \begin{bmatrix} 20/7 \\ 43/7 \\ -13/7 \\ 5/7 \end{bmatrix}$$

$$\frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-8 + 5 + 3 + 3}{4 + 1 + 1 + 1} \Rightarrow \frac{3}{7} \begin{bmatrix} -2 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6/7 \\ 3/7 \\ -3/7 \\ 3/7 \end{bmatrix}$$

$$\vec{y}_{\perp} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix} - \begin{bmatrix} 20/7 \\ 43/7 \\ -13/7 \\ 5/7 \end{bmatrix} = \begin{bmatrix} 8/7 \\ -8/7 \\ -8/7 \\ 16/7 \end{bmatrix}$$

$$\frac{\vec{x} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{4 + 5 + 6 - 3}{1 + 1 + 4 + 1} \Rightarrow \frac{12}{7} \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 12/7 \\ 12/7 \\ -24/7 \\ -12/7 \end{bmatrix}$$

which is a multiple of \vec{u}_4

\vec{y}_{\parallel} best approx.

distance = $\|\vec{y}_{\perp}\|$

$$\sqrt{\left(\frac{8}{7}\right)^2 + \left(\frac{-8}{7}\right)^2 + \left(\frac{-8}{7}\right)^2 + \left(\frac{16}{7}\right)^2} = \sqrt{\frac{64}{7}} = \frac{8}{\sqrt{7}}$$

$$b. \vec{u}_4 \cdot \vec{u}_2 = -1 + 1 + 0 = 0$$

$$\frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{-1 + 4 + 0}{1 + 1} \Rightarrow \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix}$$

$$\frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{1 + 4 + 0}{1 + 1} \Rightarrow \frac{5}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix}$$

$$\vec{y}_{\parallel} = \begin{bmatrix} 3/2 \\ 3/2 \\ 0 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 5/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \quad \vec{y}_{\perp} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \quad \text{in } \omega^{\perp}$$

best approx in ω

$$\|\vec{y}_{\perp}\| = \sqrt{3^2} = 3$$

2nd homework #7 key

(2)

2c. $\vec{u}_1 \cdot \vec{u}_2 = 1+0+0-1=0$ $\vec{u}_1 \cdot \vec{u}_3 = 0-4+0+1=0$ $\vec{u}_2 \cdot \vec{u}_3 = 0+0+1-1=0$

$\frac{\vec{x} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{3+4+0-6}{1+1+1} \Rightarrow \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ -1/3 \end{bmatrix}$ $\vec{y}_{||} = \begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5/3 \\ -7/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$

$\frac{\vec{x} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{3+0+5+6}{1+1+1} \Rightarrow \frac{14}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix}$ $\vec{y}_{\perp} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

$\frac{\vec{x} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{0-4+5-6}{1+1+1} \Rightarrow -\frac{5}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix}$

3. a. $\vec{v}_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ $\frac{\vec{v}_1 \cdot \vec{u}_2}{\vec{v}_1 \cdot \vec{v}_1} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \frac{6+5+2}{4+25+1} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \frac{13}{30} \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 13/15 \\ -13/6 \\ 13/30 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} 13/15 \\ -13/6 \\ 13/30 \end{bmatrix} = \begin{bmatrix} 49/15 \\ 1 \\ 29/30 \end{bmatrix} \left\{ \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 16 \\ 5 \\ 8 \end{bmatrix} \right\}$

b. $\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}$ $\frac{\vec{v}_1 \cdot \vec{u}_2}{\vec{v}_1 \cdot \vec{v}_1} \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \frac{-15-9-18-3}{9+1+4+1} \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \frac{-45}{15} \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 3 \\ -6 \\ 3 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} -5 \\ 9 \\ -9 \\ 3 \end{bmatrix} - \begin{bmatrix} -9 \\ 3 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix} \left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix} \right\}$

4. $A = QR \Rightarrow Q^T A = Q^T Q R = R$

$\begin{bmatrix} 5/6 & 1/6 & -3/6 & 1/6 \\ -1/6 & 5/6 & 1/6 & 3/6 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 25/6 + 1/6 + 9/6 + 1/6 & 45/6 + 7/6 + 15/6 + 5/6 \\ -5/6 + 35/6 - 3/6 + 15/6 & -5/6 + 15/6 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix} = R$

- 5. a. false UTU is
- b. true
- c. true
- d. true
- e. true
- f. false its \mathbb{R}^5 ; the range is \mathbb{R}^3 .
- g. true (if we allow for infinite dimensional matrices)