202 Proof Set #2 key

1. We want to show that if a, b real and $1<a<b$, then $1/a > 1/b$. Suppose it's not the case that $1<a<b$, i.e $0<b<a<1$. if $a, b \neq 0$ we can divide $b<a$ by $ab > 0$, then $\frac{b}{ab} < \frac{a}{ab} < \frac{1}{ab} \Rightarrow \frac{1}{a} < \frac{1}{b}$ therefore, if $1<a<b$, $\frac{1}{a} > 1/b$. //

2. to show that the zero vector in a vector space is unique, we will assume it is not unique. both $\vec{u}$ and $\vec{v}$ are the zero vector. Then we need to show that $\vec{u} = \vec{v}$. the zero vector is the additive identity so $\vec{x} + \vec{0} = \vec{x}$. if both $\vec{u}$ and $\vec{v}$ act like the zero vector, then $\vec{x} + \vec{u} = \vec{x}$ and $\vec{x} + \vec{v} = \vec{x}$. Using the property of the additive inverse, we add $-\vec{x}$ to both sides of the equation giving us $\vec{x} + \vec{v} + (-\vec{x}) = \vec{x} + (-\vec{x})$ and $\vec{x} + \vec{u} + (-\vec{x}) = \vec{x} + (-\vec{x})$ which implies $\vec{x} + (-\vec{x}) + \vec{v} = \vec{x} + (-\vec{x})$ and $\vec{x} + (-\vec{x}) + \vec{u} = \vec{x} + (-\vec{x})$ by the property of commutativity. by definition of the additive inverse we can cancel $\vec{x} + (-\vec{x}) = \vec{0}$ which implies $\vec{u} + \vec{0}$ and $\vec{v} + \vec{0} = \vec{0}$ in both cases. Thus $\vec{u} = \vec{v}$. The zero vector is not unique. //

3. a. let $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and let $C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

if we exchange two rows of A to obtain B Then $B = \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$

$\det A = a_{11}a_{22} - a_{21}a_{12}$ and $\det B = a_{21}a_{12} - a_{11}a_{22} = -(a_{11}a_{22} - a_{21}a_{12})$
$= -\det A$. //

b. if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Then by definition of the transpose, $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$\det A^T = a_{11}a_{22} - a_{12}a_{21}$. Since a_{12}, a_{21} are real numbers, $a_{11}a_{22} - a_{12}a_{21} =$
$a_{11}a_{22} - a_{21}a_{12} = \det A$. //

c. If $AC = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix}$ then

$\det(AC) = (a_{11}c_{11}+a_{12}c_{21})(a_{21}c_{12}+a_{22}c_{22}) - (a_{21}c_{11}+a_{22}c_{21})(a_{11}c_{12}+a_{12}c_{22}) =$

$\cancel{a_{11}c_{11}a_{21}c_{12}} + a_{11}c_{11}a_{22}c_{22} + a_{12}c_{21}a_{21}c_{12} + \cancel{a_{12}c_{21}a_{22}c_{22}} - \cancel{a_{21}c_{11}a_{11}c_{12}} -$

$a_{21}c_{11}a_{12}c_{22} - a_{22}c_{21}a_{11}c_{12} - \cancel{a_{22}c_{21}a_{12}c_{22}}$

on the other hand $\det A = (a_{11}a_{22} - a_{12}a_{21})$ and $\det C = (c_{11}c_{22} - c_{12}c_{21})$

So $\det A \cdot \det C = (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21}) =$

$a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21}$

after cancelling terms and comparing the remaining ones we find that

$a_{11}c_{11}a_{22}c_{22} + a_{12}c_{21}a_{21}c_{12} - a_{21}c_{11}a_{12}c_{22} - a_{22}c_{21}a_{11}c_{12} =$

$a_{11}a_{22}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21}$

Thus $\det(AC) = \det A \cdot \det C$. //

d. first, it's clear from the property above that $\det A^2 = \det(A \cdot A) = \det A \cdot \det A = (\det A)^2$. So we suppose the property is true for $\det A^k = (\det A)^k$ and show true for $k+1$. $\det(A^{k+1}) = \det(A^k \cdot A) = \det(A^k) \cdot \det A = (\det A)^k \cdot \det A = (\det A)^{k+1}$. Thus $\det(A^k) = (\det A)^k$ is true for all $k \in \mathbb{N}$. //

e. Consider r is real and $rA = \begin{bmatrix} ra_{11} & ra_{12} \\ ra_{21} & ra_{22} \end{bmatrix}$ and $\det(rA) =$

$ra_{11}ra_{22} - ra_{12}ra_{21} = r^2a_{11}a_{22} - r^2a_{12}a_{21}$ by commutativity and associativity of real #'s and by the distributive property we have $r^2(a_{11}a_{22} - a_{12}a_{21}) = r^2 \det A$.

Thus, $\det(rA) = r^2 \det A$. //