212 Homework #13 key

1a. $m_1 x_1'' = -k_1 x_1 - k_2 x_2 + k_2 x_2 \Rightarrow x_1'' = -5x_1 + 3x_2$

$m_2 x_2'' = -k_2 x_2 + k_2 x_1 \Rightarrow 5x_2'' = -3x_2 + 3x_1$

b. $m_1 x_1'' = -k_1 x_1 - k_2 x_1 + k_2 x_2 \Rightarrow x_1'' = -5x_1 + 4x_2$

$m_1 x_2'' = -k_2 x_2 - k_3 x_2 + k_2 x_1 \Rightarrow x_2'' = -5x_2 + 4x_1$

c. $m_2 x_2'' = -k_2 x_2 + k_2 x_1 \Rightarrow x_2'' = -x_2 + x_1$

$(m_1 + m_2) x_1'' = -k_1 x_1 - k_2 x_1 + k_2 x_2 \Rightarrow 2x_1'' = -5x_1 + x_2$

d. $m_1 x_1'' = -k_1 x_1 - k_2 x_1 + k_2 x_2 \Rightarrow x_1'' = -2x_1 + x_2$

$m_2 x_2'' = -k_2 x_2 - k_3 x_2 + k_2 x_1 + k_3 x_3 \Rightarrow 2x_2'' = x_1 - 2x_2 + x_3$

$m_3 x_3'' = -k_3 x_3 + k_3 x_2 + p(t) \quad 3x_3'' = x_2 - x_3 + p(t)$

2a. $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x} \quad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$(5-\lambda)(1-\lambda) + 3 = \lambda^2 - 6\lambda + 5 + 3 = 0$

$\lambda^2 - 6\lambda + 8 = 0 \quad (\lambda - 4)(\lambda - 2) = 0$

$\lambda_1 = 4, \lambda_2 = 2$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \quad x_1 = x_2, \; x_2 = x_2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \quad 3x_1 = x_2, \; x_2 = x_2 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} \Rightarrow c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 3 & -1 \end{array}\right] \quad c_1 = 7/2, \; c_2 = -3/2$

$\vec{x}(t) = \frac{7}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} - \frac{3}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$

b. $\vec{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \vec{x}$

$(-\lambda)\begin{vmatrix} -\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 2 & -\lambda \\ -1 & 2 \end{vmatrix} =$

$(-\lambda)(-\lambda)(4-\lambda) - (4-\lambda) = 0 \quad -(\lambda^3 - 4\lambda^2 - \lambda + 4) = 0$

$4\lambda^2 - \lambda^3 - 4 + \lambda = 0 \quad \lambda^2(\lambda - 4) - 1(\lambda - 4) = 0$

$(\lambda^2 - 1)(\lambda - 4) = 0$

$\lambda = \pm 1, \; \lambda = 4$

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2b cont'd.

$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = -x_3,\ x_2 = -2x_3,\ x_3 = x_3$ $\quad \vec{v}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$ $\quad \lambda_1 = 1$

$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = x_2,\ x_2 = -2x_2,\ x_3 = x_3$ $\quad \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\quad \lambda_2 = -1$

$\begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = -1/4\,x_3,\ x_2 = -1/8\,x_3,\ x_3 = x_3$ $\quad \vec{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}$ $\quad \lambda_3 = 4$

$$\vec{x}(t) = c_1\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + c_3\begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}e^{4t} \qquad \vec{x}(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -2 & 7 \\ -2 & -2 & -1 & 5 \\ 1 & 1 & 8 & 5 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = -6 \\ c_2 = 3 \\ c_3 = 1 \end{array}$$

$$\vec{x}(t) = \begin{pmatrix} 6 \\ 12 \\ -6 \end{pmatrix}e^{t} + \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}e^{-t} + \begin{pmatrix} -2 \\ -1 \\ 8 \end{pmatrix}e^{4t}$$

3a. $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\vec{x}$ $\quad (3-\lambda)(-1-\lambda) + 8 = 0$

$\lambda^2 - 2\lambda - 3 + 8 = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 20}}{2}$ $\quad$ 4pi

$= 1 \pm 2i$

$\begin{pmatrix} 3-1-2i & -2 \\ 4 & -1-1-2i \end{pmatrix} = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}$ $\quad (2-2i)x_1 = (1+i)x_2$ $\quad x_2 = \dfrac{2}{x_2}$ $\quad \begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$$e^{t}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}(\cos 2t + i\sin 2t) = e^{t}\begin{pmatrix} \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \\ 2\cos 2t + 2i\sin 2t \end{pmatrix}$$

Real part / Imaginary part

$$c_1 e^{t}\begin{pmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{pmatrix} + c_2 e^{t}\begin{pmatrix} \sin 2t + \cos 2t \\ 2\sin 2t \end{pmatrix} = \vec{x}(t)$$

use software to draw the direction field

Spirals outward