

H2 Homework #8 Key

1a. $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$ $\lim_{n \rightarrow \infty} \left| \frac{x^{2(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{2n} x^2}{(n+1)n!} \cdot \frac{n!}{x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^2}{n+1} \right| = 0$

Converges on $(-\infty, \infty)$, radius of convergence is ∞

b. $\sum_{n=0}^{\infty} 2^n x^n$ $\lim_{n \rightarrow \infty} \left| \frac{2^{n+1} x^{n+1}}{2^n x^n} \right| = \lim_{n \rightarrow \infty} |2x| < 1$
 converges on $-1 < 2x < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$ Radius of convergence $\frac{1}{2}$

c. $\sum_{n=0}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}$ $\lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 (x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 (x+2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{n^2} \right| \cdot \left| \frac{x+2}{3} \right|$
 $\lim_{n \rightarrow \infty} \left| \left(1 + \frac{1}{n}\right)^2 \right| \cdot \left| \frac{x+2}{3} \right| < 1 \Rightarrow -1 < \frac{x+2}{3} < 1 \Rightarrow \frac{-3}{-2} < \frac{x+2}{-2} < \frac{3}{-2}$

Radius of convergence is 3; converges on $-5 < x < 1$

d. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$ $\lim_{n \rightarrow \infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \cdot |x-1| < 1$
 $\frac{-1 < x-1 < 1}{+1 \quad +1 \quad +1}$ radius of convergence is 1
 converges on $(0, 2)$

2a. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

b. $\frac{1}{x+1} = \sum_{n=0}^{\infty} (-1)^n x^n$

c. $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

d. $x^3, x_0=1$

$f(x) = x^3 \quad f(1) = 1$
 $f'(x) = 3x^2 \quad f'(1) = 3 \quad \frac{f'(1)}{1!} = 3$
 $f''(x) = 6x \quad f''(1) = 6 \quad \frac{f''(1)}{2!} = \frac{6}{2} = 3$
 $f'''(x) = 6 \quad f'''(1) = 6 \quad \frac{f'''(1)}{3!} = \frac{6}{6} = 1$
 $f^{(4)}(x) = 0$
 \vdots

$x^3 = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$

$\sum_{n=1}^{\infty} a_n r^{n-1} = (1-x)^2 \sum_{n=2}^{\infty} a_n (n-1) r^{n-2} = 3(1-x)^3$
 $\sum_{n=3}^{\infty} a_n (n-1)(n-2) r^{n-3} = 6(1-x)^4 \Rightarrow \frac{1}{6} \sum_{n=0}^{\infty} a(n+3)(n+2)(n+1) r^n$

$\frac{1}{6} \cdot 3x^2 \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) x^n = \sum_{n=0}^{\infty} \frac{1}{2} (n+3)(n+2)(n+1) x^{n+2} = \frac{1}{(1-x)^4}$

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2f. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

g. $\frac{1}{1-x}, x_0 = -2 = \frac{1}{3-(x+2)} \cdot \frac{1}{3} = \frac{1/3}{1-\frac{(x+2)}{3}}$

$1-x = 1-(x+2)+k$
 $1-x-2+k$
 $-1-x+k \Rightarrow k=2$

$u = 1/3,$
 $r = \frac{x+2}{3}$

$3-(x+2) = 3-x-2 = 1-x \checkmark$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{1}{3} \left(\frac{x+2}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} (x+2)^n$

3a. $\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

b. $\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} = \sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^{n+1} = \sum_{n=1}^{\infty} (n+2)(n+1) n a_{n+2} x^n$

c. $\sum_{n=0}^{\infty} 2a_n x^{n+2} = \sum_{n=2}^{\infty} 2a_{n-2} x^n$



4a. $x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n =$
 $\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n + a_0 x^0 = \sum_{n=1}^{\infty} [n a_n + a_n] x^n + a_0$
 $= \sum_{n=1}^{\infty} a_n (n+1) x^n + a_0$



b. $x \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n$
 $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+2} =$
 $\sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n =$
 $\sum_{n=2}^{\infty} (n+1) n a_{n+1} x^n - \sum_{n=2}^{\infty} 2(n+1) a_{n+1} x^n - 2(1) a_1 x^0 + \sum_{n=2}^{\infty} a_{n-2} x^n =$
 $+ (2)(1) a_2 x \quad - 2(2) a_2 x$

$\sum_{n=2}^{\infty} [(n+1)n a_{n+1} - 2(n+1) a_{n+1} + a_{n-2} x^n] - 2a_1 + (2a_2 - 4a_2)x$
 $= -2a_2$