

212 Homework #9 Key

1a. $y'' - y = 0 \quad x_0 = 0$ $y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [a_{n+2} (n+2)(n+1) - a_n] x^n = 0$$

Solve for $a_{n+2} \Rightarrow a_{n+2} = \frac{a_n}{(n+2)(n+1)}$

$n=0 \quad a_2 = \frac{a_0}{2 \cdot 1} = \frac{a_0}{2!}$ $n=1 \quad a_3 = \frac{a_1}{2 \cdot 3} = \frac{a_1}{3!}$ $n=2 \quad a_4 = \frac{a_2}{3 \cdot 4} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_0}{4!}$ $n=3 \quad a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{a_1}{5!}$

$n=4 \quad a_6 = \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!}$ $n=5 \quad a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7!}$

$$y = a_0 \left(1 + \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \frac{1}{6!} x^6 + \dots \right) + a_1 \left(x + \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + \dots \right)$$

$$= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = a_0 \cosh x + a_1 \sinh x \quad \text{3a.}$$

b. $y'' + 4y' + 4y = 0 \quad x_0 = 0$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + 4 \sum_{n=1}^{\infty} a_n n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=0}^{\infty} 4 a_{n+1} (n+1) x^n + \sum_{n=0}^{\infty} 4 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} [a_{n+2} (n+2)(n+1) + 4 a_{n+1} (n+1) + 4 a_n] x^n = 0$$

$$a_{n+2} = \frac{-4 a_{n+1} (n+1) - 4 a_n}{(n+2)(n+1)} = -\frac{4 a_{n+1}}{n+2} - \frac{4 a_n}{(n+2)(n+1)}$$

$n=0 \quad a_2 = -\frac{4 a_1}{2} - \frac{4 a_0}{2 \cdot 1} = -\frac{1}{2} a_1 - \frac{1}{2} a_0$

$n=1 \quad a_3 = -\frac{4 a_2}{3} - \frac{4 a_1}{3 \cdot 2} = -\frac{4}{3} \left(-\frac{1}{2} a_1 - \frac{1}{2} a_0 \right) - \frac{2}{3} a_1 = \frac{2}{3} a_1 + \frac{2}{3} a_0 - \frac{2}{3} a_1 = \frac{2}{3} a_0$

$n=2 \quad a_4 = -\frac{4 a_3}{4} - \frac{4 a_2}{4 \cdot 3} = -\left(\frac{2}{3} a_0 \right) - \frac{1}{3} \left(-\frac{1}{2} a_1 - \frac{1}{2} a_0 \right) = -\frac{2}{3} a_0 + \frac{1}{6} a_1 + \frac{1}{6} a_0 = -\frac{1}{2} a_0 + \frac{1}{6} a_1$

$n=3 \quad a_5 = -\frac{4 a_4}{5} - \frac{4 a_3}{5 \cdot 4} = -\frac{4}{5} \left(-\frac{1}{2} a_0 + \frac{1}{6} a_1 \right) - \frac{1}{5} \left(\frac{2}{3} a_0 \right) = \frac{2}{5} a_0 - \frac{1}{15} a_1 - \frac{2}{15} a_0 = \frac{4}{15} a_0 - \frac{1}{15} a_1$

$$y = a_0 \left(1 - \frac{1}{2} x^2 - \frac{2}{3} x^3 - \frac{1}{2} x^4 + \frac{4}{15} x^5 + \dots \right) + a_1 \left(x - \frac{1}{2} x^2 + \frac{1}{6} x^4 - \frac{1}{15} x^5 + \dots \right)$$

3c. $y'' + 4y' + 4y = 0$
 $r^2 + 4r + 4 = 0$
 $(r+2)^2 = 0 \quad r = -2$

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$