

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (2 points each)

- a. T F Geometrically, if λ is an eigenvalue of a matrix A , and \vec{x} is an eigenvector of A corresponding to λ , then multiplying \vec{x} by A produces a vector $\lambda\vec{x}$ parallel to \vec{x} .
- b. T F If A is an $n \times n$ matrix with an eigenvalue λ , then the set of all eigenvectors of λ is a subspace of R^n .
- c. T F The fact that an $n \times n$ matrix A has n distinct eigenvalues does not guarantee that A is diagonalizable. *it does guarantee it*
- d. T F When $P^{-1} = P^T$ then P contains an orthonormal basis for R^n . *only if P is also square*
- e. T F An eigenvector for A may be the zero vector.
- f. T F Similar matrices may or may not have the same eigenvalues. *They must*
- g. T F If one of the eigenvalues of A is 3, then one of the eigenvalues of $A - 5I$ is -2 .
- h. T F The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible when $ab - dc \neq 0$. *ad-bc \neq 0*
- i. T F The zero matrix is an elementary matrix. *not a row operation*
- j. T F Cramer's Rule can only be applied when the coefficient matrix of the system is nonsingular.
- k. T F The set of all ordered triples (x, y, z) of real numbers where $yz \geq 0$ with the standard operations on R^3 is a vector space. *fails scalar mult. rule*
- l. T F Elementary row operations preserve the column space of the matrix A . *preserve row space*
- m. T F A nonzero vector in an inner product space can have a norm of zero.
- n. T F The dimension of $M_{3 \times 4}$ is seven. *twelve*
- o. T F The set of vectors $\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

- p. T F The dimension of a vector space is equal to the number of vectors in any basis for the space.
- q. T F The orthogonal complement of R^n is the zero vector.
- r. T F An orthonormal basis derived by the Gram-Schmidt orthonormalization process does depend on the order of the vectors in the basis.
- s. T F If a set of vectors S is linearly independent, then S is also orthogonal.
- t. T F Two matrices that represent the same linear transformation $T: V \rightarrow W$ with respect to different bases are symmetric.

orthogonal \rightarrow independence

similar

2. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$. (8 points)

$$(-2-\lambda)(2-\lambda) - 1 = 0$$

$$\lambda^2 - 4 - 1 = 0$$

$$\lambda^2 - 5 = 0$$

$$\lambda = \pm\sqrt{5}$$

$$\begin{bmatrix} -2 + \sqrt{5} & 1 \\ 1 & 2 + \sqrt{5} \end{bmatrix}$$

$$\begin{aligned} x_1 &= -(2 + \sqrt{5})x_2 \\ x_2 &= x_2 \end{aligned}$$

$$\lambda_2 = \sqrt{5}$$

$$\vec{v}_1 = \begin{bmatrix} 2 + \sqrt{5} \\ -1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 2 - \sqrt{5} \\ -1 \end{bmatrix}$$

3. For the matrix $A = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}$ with eigenvalues $\lambda_1 = -1, \lambda_2 = 1$ with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the similarity transformation that diagonalizes A and find D . (6 points)

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{3-1} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

4. Use the information from #3, to find e^A . (6 points)

$$e^D = \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-1} & 0 \\ 0 & e \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} =$$

$$\begin{bmatrix} 3e+0 & 0+e \\ 1e+0 & 0+e \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e - \frac{1}{2}e & -\frac{3}{2}e + \frac{3}{2}e \\ \frac{1}{2}e - \frac{1}{2}e & -\frac{1}{2}e + \frac{3}{2}e \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{-1} - \frac{1}{2}e & -\frac{3}{2}e^{-1} + \frac{3}{2}e \\ \frac{1}{2}e^{-1} - \frac{1}{2}e & -\frac{1}{2}e^{-1} + \frac{3}{2}e \end{bmatrix} = e^A$$

5. Solve the linear system of ODEs given by $\vec{y}' = \begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix} \vec{y}$. Write the solution in standard form, and plot several sample trajectories. (10 points)

$$(1-\lambda)(1-\lambda) - 9 = 0$$

$$\lambda^2 - 2\lambda + 1 - 9 = 0$$

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = 4, -2$$

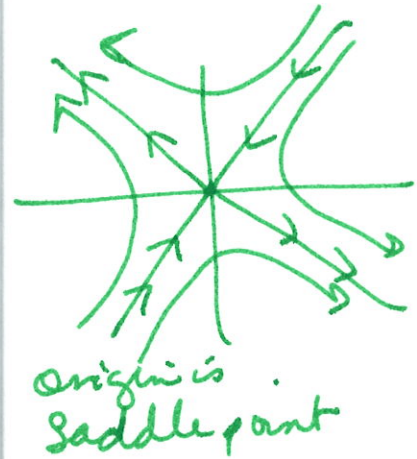
$$\lambda_1 = 4$$

$$\begin{bmatrix} -3 & -3 \\ -3 & -3 \\ 1 & 1 \end{bmatrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -2$$

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \\ 1 & -1 \end{bmatrix} \quad \begin{matrix} x_1 = x_2 \\ x_2 = x_2 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$



6. Find the LU factorization of $A = \begin{bmatrix} 1 & 7 \\ 2 & 20 \end{bmatrix}$. (6 points)

$$-2R_1 + R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = E_1$$

$$\Rightarrow \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_1^{-1}$$

$$U = \begin{bmatrix} 1 & 7 \\ 0 & 6 \end{bmatrix}$$

7. Let M be a 9×5 matrix. If M has 5 pivots, find the following: (3 points each)

a. $\dim \text{Col } A$

5

b. $\text{rank } A$

5

c. $\dim \text{Nul } A$

0

d. nullity of A

0

e. $\text{Col } A$ is a subspace of R^m . $m = ?$

9

f. $\text{Nul } A$ is a subspace of R^n . $n = ?$

5

8. Use properties of determinants and $\det(A) = -2$, $\det(B) = 4$, to find the following, given that both A and B are 4×4 matrices. (3 points each)

a. $\det(B^{-1}) = \frac{1}{4}$

b. $\det(A^{-1}B^3) = -\frac{1}{2}(4^3) = -32$

c. $\det(5A) = 5^4(-2) = -1250$

d. $\det(A^{-1}BA) = -\frac{1}{2}(4)(-2) = 4$

e. $\det(-B^T) = (-1)^4(4) = 4$