

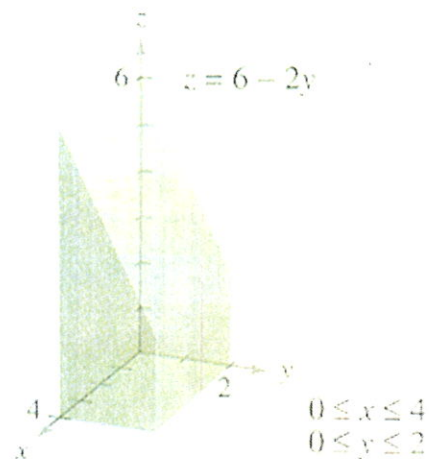
Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the volume of the solid under the function $f(x, y) = 6 - 2y$ on the region bounded by $0 \leq x \leq 4, 0 \leq y \leq 2$ using a double integral. (9 points)

$$\int_0^4 \int_0^2 6 - 2y \, dy \, dx =$$

$$\int_0^4 6y - y^2 \Big|_0^2 \, dx = \int_0^4 12 - 4 \, dx =$$

$$\int_0^4 8 \, dx = 8x \Big|_0^4 = \boxed{32}$$



2. The integral $\int_0^1 \int_{y^2}^1 \sqrt{x} \sin x \, dx \, dy$ can be evaluated only by changing the order of integration. Sketch the region of integration, reverse the order of integration, and evaluate the integral. (8 points)

$$\int_0^1 \int_{y^2}^1 \sqrt{x} \sin x \, dx \, dy =$$

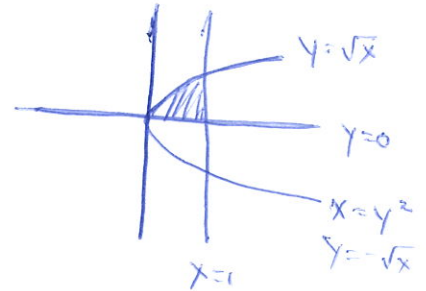
$$\int_0^1 (\sqrt{x} \sin x) \Big|_{y^2}^1 dx =$$

$$\int_0^1 \sqrt{x} \sin x \cdot \sqrt{x} \, dx = \int_0^1 x \sin x \, dx$$

$$-x \cos x \Big|_0^1 - \int_0^1 -\cos x \, dx =$$

$$-x \cos x + \sin x \Big|_0^1 = -\cos 1 + \sin 1 - 0 =$$

$$\boxed{\sin 1 - \cos 1}$$

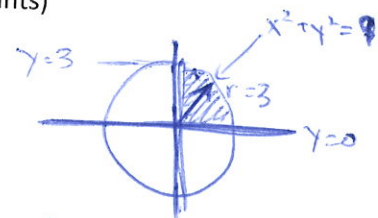


$$u = x \quad dv = \sin x \, dx$$

$$du = dx \quad v = -\cos x$$

3. Evaluate $\int_0^3 \int_0^{\sqrt{9-y^2}} x \, dx \, dy$ by converting to polar coordinates. (8 points)

$$x = r \cos \theta$$

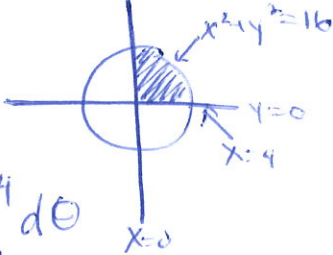


$$\int_0^{\pi/2} \int_0^3 r \cos \theta \, r \, dr \, d\theta = \int_0^{\pi/2} \int_0^3 r^2 \cos \theta \, dr \, d\theta =$$

$$\int_0^{\pi/2} \frac{1}{3} r^3 \cos \theta \Big|_0^3 \, d\theta = \int_0^{\pi/2} 9 \cos \theta \, d\theta = 9 \sin \theta \Big|_0^{\pi/2} = \boxed{9}$$

4. Evaluate the integral $\int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{x^2+y^2}} \frac{1}{1+x^2+y^2} dz dy dx$ in cylindrical coordinates. (10 points)

$y = \sqrt{16-x^2}$
 $z = \sqrt{x^2+y^2} \Rightarrow z=r$
 $x^2+y^2=r^2$ $\frac{1}{1+x^2+y^2} = \frac{1}{1+r^2}$
 $x^2+y^2=16$



$$\int_0^{\pi/2} \int_0^4 \int_0^r \frac{1}{1+r^2} r dz dr d\theta =$$

$$\int_0^{\pi/2} \int_0^4 \frac{r z}{1+r^2} \Big|_0^r dr d\theta = \int_0^{\pi/2} \int_0^4 \frac{r^2}{1+r^2} dr d\theta$$

$$\int_0^{\pi/2} \int_0^4 \left(1 - \frac{1}{1+r^2}\right) dr d\theta = \int_0^{\pi/2} \left(r - \arctan r \Big|_0^4 \right) d\theta$$

$$= \int_0^{\pi/2} (4 - \arctan 4) d\theta = (4 - \arctan 4) \theta \Big|_0^{\pi/2} =$$

$$2\pi - \frac{\pi \arctan 4}{2}$$

5. Set up an iterated integral to find the area of the region bounded by $5x - 3y = 0$, $x + y = 4$, $y = 0$. Evaluate the integral to find the area. (10 points)

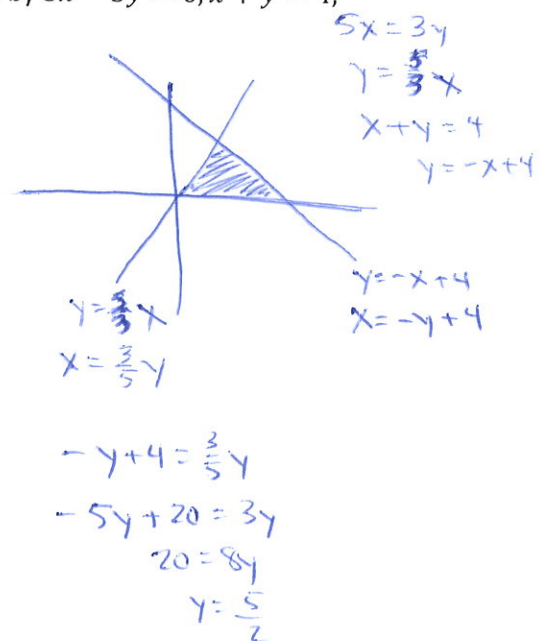
$$\int_0^{\pi/2} \int_{\frac{3}{5}y}^{-y+4} 1 dx dy =$$

$$\int_0^{\pi/2} y \Big|_{\frac{3}{5}y}^{-y+4} dy =$$

$$\int_0^{\pi/2} -y + 4 - \frac{3}{5}y dy = \int_0^{\pi/2} -\frac{8}{5}y + 4 dy$$

$$= -\frac{4}{5}y^2 + 4y \Big|_0^{\pi/2} = -\frac{4}{5} \left(\frac{\pi}{2}\right)^2 + 4\left(\frac{\pi}{2}\right) - 0 =$$

$$-5 + 10 = \boxed{5}$$



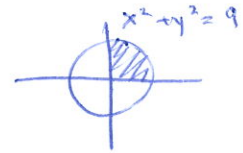
6. Use a double integral to find the volume of the indicated solid: $z = x + y, x^2 + y^2 = 9$, first octant. (10 points)

$$\int_0^{\pi/2} \int_0^3 r(\cos\theta + \sin\theta) r dr d\theta =$$

$$\int_0^{\pi/2} \frac{1}{3} r^3 (\cos\theta + \sin\theta) \Big|_0^3 d\theta =$$

$$9 \int_0^{\pi/2} (\cos\theta + \sin\theta) d\theta = 9 \left[\sin\theta - \cos\theta \Big|_0^{\pi/2} =$$

$$9 [1 - 0 - (0 - 1)] = \boxed{18}$$



$$x + y = r\cos\theta + r\sin\theta$$

$$= r(\cos\theta + \sin\theta)$$

7. Find the area of the surface given by $f(x, y) = 5 + 2x - 2y$ over the region R : rectangle with vertices $(0,0), (3,0), (3,1), (0,1)$. (10 points)

$$\underbrace{0 \leq x \leq 3} \quad \underbrace{0 \leq y \leq 1}$$

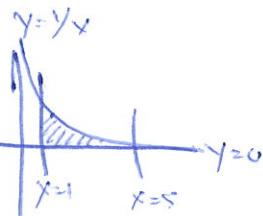
$$\int_0^3 \int_0^1 3 dy dx =$$

$$\begin{aligned} f_x &= 2 \\ f_y &= -2 \\ \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + 4 + 4} = \sqrt{9} = 3 \end{aligned}$$

$$\int_0^3 3y \Big|_0^1 dx = \int_0^3 3 dx = 3x \Big|_0^3 = \boxed{9}$$

8. Find the mass and centroid of the center of mass of the lamina bounded by the graphs of the equations $xy = 1$, $x = 1$, $x = 5$, $y = 0$ with the density $\rho = kxy$. (15 points)

$$y = \frac{1}{x}$$



$$M_z = \int_1^5 \int_0^{1/x} kxy \, dy \, dx = \int_1^5 \left. \frac{k}{2} xy^2 \right|_0^{1/x} dx$$

$$= \int_1^5 \frac{k}{2} \cdot \frac{1}{x} dx = \frac{k}{2} \ln x \Big|_1^5 = \frac{k}{2} \ln 5$$

$$M_y = \int_1^5 \int_0^{1/x} kx^2 y \, dy \, dx = \int_1^5 \left. \frac{k}{2} x^2 y^2 \right|_0^{1/x} dx = \int_1^5 \frac{k}{2} dx = \frac{k}{2} x \Big|_1^5 = \frac{k}{2} (5-1) = 2k$$

$$M_x = \int_1^5 \int_0^{1/x} kxy^2 \, dy \, dx = \int_1^5 \left. \frac{k}{3} xy^3 \right|_0^{1/x} dx = \int_1^5 \frac{k}{3} x^{-2} dx = -\frac{k}{3} \frac{1}{x} \Big|_1^5 = -\frac{k}{3} \cdot \frac{1}{5} + \frac{k}{3} \cdot \frac{1}{1} = \frac{k}{3} - \frac{k}{15} = \frac{4}{15}k$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{4}{15}k}{\frac{k}{2} \ln 5} = \frac{4}{15} \cdot 2 \cdot \frac{1}{\ln 5} = \frac{8}{15 \ln 5}$$

$$\bar{x} = \frac{M_y}{M} = \frac{2k}{\frac{k}{2} \ln 5} = \frac{4}{\ln 5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{\ln 5}, \frac{8}{15 \ln 5} \right)$$

9. Find integrals for I_x, I_y, I_z for the cube $Q: \{(x, y, z): 0 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq 4\}$ with density $\rho(x, y, z) = kxyz$. You do not need to integrate them. (10 points)

$$I_x = \int_0^2 \int_0^3 \int_0^4 kxyz (y^2 + z^2) dz dy dx$$

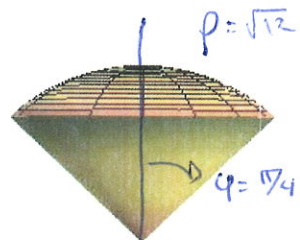
$$I_y = \int_0^2 \int_0^3 \int_0^4 kxyz (x^2 + z^2) dz dy dx$$

$$I_z = \int_0^2 \int_0^3 \int_0^4 kxyz (x^2 + y^2) dz dy dx$$

10. Find the volume of the solid between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 12$. (10 points)

$$\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sqrt{12}} \rho^2 \sin \varphi d\rho d\theta d\varphi$$

$$\rho \cos \varphi = \rho \sin \varphi \Rightarrow \varphi = \pi/4$$



$$= \int_0^{\pi/4} \int_0^{2\pi} \frac{\rho^3}{3} \Big|_0^{\sqrt{12}} \sin \varphi d\theta d\varphi = \int_0^{\pi/4} \int_0^{2\pi} \frac{12\sqrt{12}}{3} \sin \varphi d\theta d\varphi$$

$$= \int_0^{\pi/4} 4\sqrt{12} \sin \varphi \cdot \theta \Big|_0^{2\pi} d\varphi = \int_0^{\pi/4} 8\pi\sqrt{12} \sin \varphi d\varphi =$$

$$-8\pi\sqrt{12} \cos \varphi \Big|_0^{\pi/4} = -8\pi\sqrt{12} \left(\frac{1}{\sqrt{2}} - 1 \right) = \frac{16\pi\sqrt{3}(\sqrt{2}-1)}{\sqrt{2}}$$